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On Conformal Curves in 2-Dimensional de Sitter Space

Hakan Simsek and Mustafa Özdemir

Abstract. In this paper, we examine the pseudo-spherical curves, which are equivalent to each other under the conformal maps preserving a fixed point in the de Sitter 2-space, by using the Clifford algebra $Cl_{2,1}$. Also, we find the parametric equations of de Sitter loxodromes.

Mathematics Subject Classification. 14H50, 53A35, 53B30, 53C50.

Keywords. De Sitter space · Loxodrome · Clifford algebras.

1. Introduction

A 2-dimensional de Sitter space \mathcal{S}^2 is a Lorentzian manifold analog, embedded in Minkowski space $\mathcal{M}^{2,1}$, of the Euclidean sphere. It is maximally symmetric, has a positive constant curvature, and it corresponds to a one-sheeted hyperboloid which is given by

$$\mathcal{S}_r^2 = \{(u_1, u_2, u_3) \in \mathcal{M}^{2,1} : -u_1^2 + u_2^2 + u_3^2 = r^2, r \in \mathbb{R}\}$$

with the signature $(-, +, +)$. The de Sitter space is named after Willem de Sitter (1872–1934), professor of astronomy at Leiden University [6, 17].

The de Sitter space has a physical importance in the view of relativity theory. It is the vacuum solution of Einstein's field equations with a positive cosmological constant that exhibits maximal symmetry [18]. It was the first interacting quantum field theory constructed on a curved space-time, the so-called $\mathcal{P}(\varphi)_2$ model on the de Sitter 2-space [3]. Also, the problem of localizability related to the quantum field theory was investigated in \mathcal{S}^2 by [20].

The (Clifford) geometric algebras are a type of associative algebras. They are a powerful and practical framework for the representation and solution of geometrical problems. We can think of them as a structure generalizing the hypercomplex number systems such as the complex numbers, quaternions, split quaternions, double numbers. Geometric algebras have important applications in a variety of fields including geometry, kinematics, theoretical physics and digital image processing. They are named after the English

geometer William Kingdon Clifford. The most important Clifford algebras are those over real and complex vector spaces equipped with nondegenerate quadratic forms.

The Loxodromes, also known as a rhumb line, are a path on Earth, which cuts all meridians of longitude at any constant angle. It is a straight line on a Mercator projection map and can be drawn on such a map between any two points on Earth without going off the edge of the map. The loxodromes are not the shortest distance between two points on a sphere. Near the poles, they are close to being logarithmic spirals (see [1, 8, 19]).

Encheva and Georgiev [7] studied some classes of curves on the shape sphere by using a special conformal map between the two-dimensional sphere and the extended plane. Babaarslan and Munteanu [2] examined the time-like loxodromes on rotational surfaces in $\mathcal{M}^{2,1}$.

The content of paper is as follows. We give some basic knowledges about Clifford algebra $Cl_{2,1}$ and study the some properties of Lorentzian plane curves in $Cl_{2,1}$. Using the powerful methods of Clifford algebra, we find a special conformal transformation between a de Sitter 2-space and the extended Minkowski plane such that we classify the pseudo-spherical curves on de Sitter 2-space by means of this special conformal transformation. Also, we examine de Sitter loxodromes which are the images of hyperbolic logarithmic spirals under the inverse generalized stereographic projection.

2. Preliminaries

The Clifford algebra $Cl_{p,q}$ is an associative and distributive geometric algebra generated by a pseudo-Euclidean vector space $\mathcal{M}^{p,q}$ equipped with a quadratic form Q . The algebra operation $\mathbf{x}\mathbf{y}$, called the *geometric product*, for any $\mathbf{x}, \mathbf{y} \in \mathcal{M}^{p,q}$ is defined by

$$\begin{aligned}\mathbf{x}\mathbf{x} &= \mathbf{x}^2 = Q(\mathbf{x}), \\ \mathbf{x}\mathbf{y} &= \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \wedge \mathbf{y}\end{aligned}$$

where $\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \wedge \mathbf{y}$ are inner product and outer product of $\mathcal{M}^{p,q}$ and $Q(\mathbf{x}) = -\sum_{t=1}^q x_t^2 + \sum_{t=q+1}^{p+q} x_t^2$ for $\mathbf{x} = (x_1, \dots, x_{p+q})$. We can express the inner product and outer product in terms of the geometric product:

$$\begin{aligned}\mathbf{x} \cdot \mathbf{y} &= \frac{1}{2}(\mathbf{x}\mathbf{y} + \mathbf{y}\mathbf{x}) \\ \mathbf{x} \wedge \mathbf{y} &= \frac{1}{2}(\mathbf{x}\mathbf{y} - \mathbf{y}\mathbf{x}).\end{aligned}$$

In this paper, we shall deal with the Clifford algebra $Cl_{2,1} = \text{gen}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ defined by the geometric product rules

$$\begin{aligned}\mathbf{i}^2 &= -1 \text{ and } \mathbf{j}^2 = \mathbf{k}^2 = 1 \\ \mathbf{ij} &= \mathbf{i} \wedge \mathbf{j} = -\mathbf{ji}, \mathbf{ik} = \mathbf{i} \wedge \mathbf{k} = -\mathbf{ki} \quad \text{and} \quad \mathbf{jk} = \mathbf{j} \wedge \mathbf{k} = -\mathbf{kj}\end{aligned}$$

where $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is the standard basis of Minkowski 3-vector space $\mathcal{M}^{2,1}$. Letting $I := \mathbf{ijk}$, any element of $Cl_{2,1}$, called a *multivector* or *geometric number*, has the form

$$s + tI + \mathbf{x} + I\mathbf{y},$$

where $s, t \in \mathbb{R}$ and $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, $\mathbf{y} = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$ for $x_l, y_l \in \mathbb{R}$, $l = 1, 2, 3$. In other words, the multivectors in $Cl_{2,1}$ are linear combinations of scalars (0-vector) s , vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (1-vector), bivectors (2-vector) $\mathbf{ij}, \mathbf{ik}, \mathbf{jk}$ and trivector (3-vector) \mathbf{ijk} . The nondivision algebra of split quaternions is isomorphic with the even subalgebra $Cl_{2,1}^+$ of the Clifford algebra $Cl_{2,1}$ where $Cl_{2,1}^+$ has the basis $\{1, \mathbf{jk}, \mathbf{ki}, \mathbf{ij}\}$. One can find more information about the Clifford algebras in [10, 11, 15].

We can study the Minkowski 3-vector space $\mathcal{M}^{2,1}$ and Minkowski plane $\mathcal{M}^{1,1}$, which is a sub-manifold of $\mathcal{M}^{2,1}$, by means of the Clifford algebra $Cl_{2,1}$ by defining as the following

$$\begin{aligned} \mathcal{M}^{2,1} &= \{\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} : x_1, x_2, x_3 \in \mathbb{R}\} \text{ and} \\ \mathcal{M}^{1,1} &= \{x_1\mathbf{i} + x_2\mathbf{j} : x_1, x_2 \in \mathbb{R}\}, \end{aligned}$$

respectively. The vector \mathbf{x} is called a spacelike vector, lightlike (or null) vector and timelike vector if $\mathbf{x}^2 > 0$ or $\mathbf{x} = 0$, $\mathbf{x}^2 = 0$ or $\mathbf{x}^2 < 0$, respectively. The norm of the vector \mathbf{x} is described by $\|\mathbf{x}\| = \sqrt{|\mathbf{x}^2|}$. Also, the inverse of any nonnull vector \mathbf{x} can be defined in the Clifford algebra as the following

$$\mathbf{x}^{-1} = \frac{\mathbf{x}}{\mathbf{x}^2}.$$

The Lorentzian vector cross product $\mathbf{x} \times \mathbf{y}$ is given by

$$\mathbf{x} \times \mathbf{y} = I(\mathbf{x} \wedge \mathbf{y}) = \det \begin{pmatrix} -\mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

The equation $\sqrt{|\mathbf{w}^2|} = a > 0$ in $\mathcal{M}^{1,1}$ implies a four branched hyperbola of hyperbolic radius a . The vector $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j}$ can be written

$$\mathbf{w} = \pm a (\mathbf{i} \cosh \theta + \mathbf{j} \sinh \theta) = \pm a \mathbf{i} e^{J\theta}$$

when \mathbf{w} lies in the hyperbolic quadrants $H-I$ or $H-III$, or

$$\mathbf{w} = \pm a (\mathbf{i} \sinh \theta + \mathbf{j} \cosh \theta) = \pm a \mathbf{j} e^{J\theta}$$

when \mathbf{w} lies in the hyperbolic quadrants $H-II$ or $H-IV$, respectively, where $J = \mathbf{ji}$. Each of the four hyperbolic branches is covered exactly once, in the indicated directions, as the parameter θ increases, $-\infty < \theta < \infty$ (See Fig. 1). The hyperbolic angle θ is called *argument* of \mathbf{w} and denoted by $\arg(\mathbf{w}) = \theta$.

The hyperbolic angle can be defined by $\tanh^{-1}(w_2/w_1)$ in the quadrants $H-I$ and $H-III$, or $\tanh^{-1}(w_1/w_2)$ in $H-II$ and $H-IV$, respectively.

The Lorentzian rotation in $\mathcal{M}^{1,1}$ can be expressed with a *spinor*, is a linear combination of a scalar and a bivector. If we take any vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{B} = \mu_1 + \mu_2 J$, then the geometric product of \mathbf{v} and \mathbf{B} is equal to

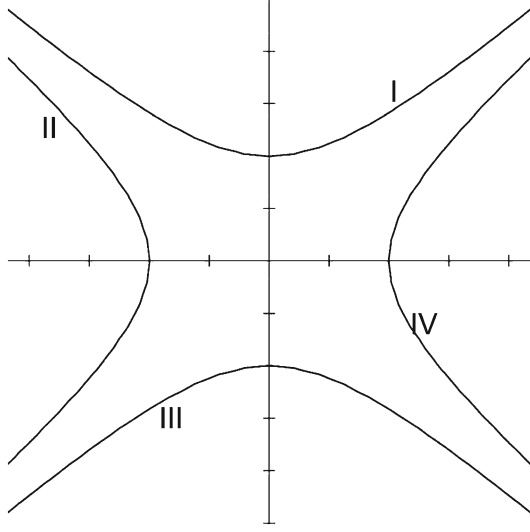


FIGURE 1. 2-hyperbola

$$\mathbf{vB} = (v_1\mu_1 + v_2\mu_2)\mathbf{i} + (v_1\mu_2 + v_2\mu_1)\mathbf{j} = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

which is a vector in $\mathcal{M}^{1,1}$. When $\mu_1 = \cosh \theta$ and $\mu_2 = \sinh \theta$, the spinor has the form $\mathbf{B} = \cosh \theta + \sinh \theta J = e^{\theta J}$ and \mathbf{vB} is a vector obtained by rotation of \mathbf{v} through θ . The geometric product of two spinor gives a new spinor. Thus, the spinors form a subgroup of $Cl_{2,1}$.

The set of extended Minkowski plane $\tilde{\mathcal{M}}^{1,1}$ is the union of the sets $\mathcal{M}^{1,1}$ and I_∞ given by

$$I_\infty = \left\{ (p\mathbf{i} \pm p\mathbf{j})^{-1} : p \in \mathbb{R} \cup \{\infty\} \right\}.$$

We state the points in I_∞ as the points at infinity. The set I_∞ can be considered as two lines at infinity that intersect at $(0\mathbf{i} + 0\mathbf{j})^{-1}$.

In $\tilde{\mathcal{M}}^{1,1}$, the equation of any pseudo-circle \mathcal{P} can be written as

$$A\mathbf{w}^2 + 2\mathbf{B} \cdot \mathbf{w} + C = 0 \quad (1)$$

or

$$\left(\mathbf{w} + \frac{\mathbf{B}}{A} \right)^2 = \frac{\mathbf{B}^2 - AC}{A^2} \quad (2)$$

where $A, C \in \mathbb{R}$, $\frac{\mathbf{B}^2 - AC}{A^2} \neq 0$ and $\mathbf{B} \in \mathcal{M}^{1,1}$. From here, $\frac{-\mathbf{B}}{A}$ is the centre of the pseudo-circle and $\left| \frac{\mathbf{B}^2 - AC}{A^2} \right|$ is the square of the radius of pseudo-circle in $\tilde{\mathcal{M}}^{1,1}$. A pseudo-circle also contains point(s) at infinity. These points in I_∞ are given by

$$\begin{aligned} \text{i) } (p_1 \mathbf{i} + p_1 \mathbf{j})^{-1} \text{ where } p_1 &= \begin{cases} \frac{A}{b_1 + b_2} & \text{if } -b_1 \neq b_2 \\ \infty & \text{if } -b_1 = b_2 \end{cases} \\ \text{ii) } (p_2 \mathbf{i} - p_2 \mathbf{j})^{-1} \text{ where } p_2 &= \begin{cases} \frac{-A}{-b_1 + b_2} & \text{if } b_1 \neq b_2 \\ \infty & \text{if } b_1 = b_2 \end{cases} \end{aligned}$$

where $\mathbf{B} = b_1 \mathbf{i} + b_2 \mathbf{j}$ and notice that $(\infty \mathbf{i} + \infty \mathbf{j})^{-1} \neq (\infty \mathbf{i} - \infty \mathbf{j})^{-1}$. If $A \neq 0$, the pseudo-circle contains definitely two points at infinity. But, if A is equal to zero, then \mathcal{P} is a line and only contains one point at infinity (see [9, 13] for double numbers). Also, \mathcal{P} is a line if and only if $(0\mathbf{i} + 0\mathbf{j})^{-1} \in \mathcal{P}$.

Now, we examine a *direct linear-fractional (or Möbius) transformation* of $\tilde{\mathcal{M}}^{1,1}$, which are mappings $T : \tilde{\mathcal{M}}^{1,1} \rightarrow \tilde{\mathcal{M}}^{1,1}$ defined by

$$T(\mathbf{w}) = (\mathbf{iaw} + \mathbf{b})(\mathbf{icw} + \mathbf{d})^{-1} \mathbf{i}$$

respectively, where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathcal{M}^{1,1}$ and $\mathbf{iad} - \mathbf{bic} \neq p\mathbf{i} \pm p\mathbf{j}$ for $p \in \mathbb{R}$. In case of $\mathbf{iad} - \mathbf{bic} \neq p\mathbf{i} \pm p\mathbf{j}$, the Möbius transformation maps all Minkowski plane to a single point or the lines have slope ± 1 . The set of these transformations form a group under the operation of composition.

The linear fractional transformation is a composition of affine transformations $\mathbf{w} \rightarrow \mathbf{iaw} + \mathbf{b}$ and multiplicative inversion $\mathbf{w} \rightarrow 1/\mathbf{w}$. The conformality of this map can be confirmed by showing its components are all conformal. Therefore, the linear fractional transformations are conformal and bijective maps in $\tilde{\mathcal{M}}^{1,1}$. Moreover, if we assume that a line is pseudo-circle which its radius is infinite, this transformation maps a pseudo-circle to another pseudo-circle. If the pseudo-circle (1) pass through the point $\mathbf{c}^{-1}\mathbf{id}$, its image becomes a line. The image of pseudo-circle under the linear-fractional transformation $\eta = T(\mathbf{w})$ can be given by

$$(-A\mathbf{d}^2 - 2\mathbf{icd} \cdot \mathbf{B} + C\mathbf{c}^2)\eta^2 + 2(A\mathbf{ibd} + -\mathbf{iadiB} + \mathbf{biciB} - C\mathbf{aic}) \cdot \eta \quad (3)$$

$$-A\mathbf{b}^2 - 2\mathbf{iab} \cdot \mathbf{B} - C\mathbf{a}^2 = 0.$$

3. Analysing of Lorentzian Plane Curves Via the Hyperbolic Structure

We define the hyperbolic structure on the Lorentzian plane, which is essential implement in order to examine the differential geometry of curves. The *hyperbolic structure* of $\mathcal{M}^{1,1}$ is the linear map $\mathcal{J} : \mathcal{M}^{1,1} \rightarrow \mathcal{M}^{1,1}$ given by

$$\mathcal{J}\mathbf{x} = \mathbf{xij} = (x_1 \mathbf{i} + x_2 \mathbf{j}) \mathbf{ij} = -x_2 \mathbf{i} - x_1 \mathbf{j}, \quad \text{for any } \mathbf{x} = x_1 \mathbf{i} + x_2 \mathbf{j}. \quad (4)$$

This is equivalent to multiplying $z(-i)$, rotating z counterclockwise by 90° in the complex number plane and called complex structure of Euclidean plane. It is easy to prove that the hyperbolic structure has the following properties

$$\begin{aligned} \mathcal{J}^2 &= \mathbf{I}, \\ (\mathcal{J}\mathbf{x}) \cdot (\mathcal{J}\mathbf{y}) &= -\mathbf{x} \cdot \mathbf{y}, \\ \mathcal{J}\mathbf{x} \cdot \mathbf{x} &= 0, \\ \mathbf{xy} &= \mathbf{x} \cdot \mathbf{y} + (\mathbf{x} \cdot \mathcal{J}\mathbf{y}) \mathbf{ij} \end{aligned} \quad (5)$$

for $\mathbf{x}, \mathbf{y} \in \mathcal{M}^{1,1}$ where $\mathbf{I} : \mathcal{M}^{1,1} \rightarrow \mathcal{M}^{1,1}$ is the identity linear map. Also, the matrix representation of the hyperbolic structure can be given by $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Therefore, we can state (4) via the matrix representation as

$$\begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -x_2 & -x_1 \\ -x_1 & -x_2 \end{bmatrix}.$$

In the rest of the paper, we will show the hyperbolic structure with \mathcal{J} .

Let's consider a smooth and regular non-lightlike curve $\gamma : U \rightarrow \mathcal{M}^{1,1}$

$$\gamma(s) = \gamma_1(s) \mathbf{i} + \gamma_2(s) \mathbf{j}$$

parameterized by arc length s , where U is a open interval in \mathbb{R} . Let's denote by $\varphi(s)$ the hyperbolic angle between the tangent vector at a point and the positive direction. The *curvature* at a point measures the rate of bending as the point moves along the curve with unit speed and can be defined as

$$\kappa(s) = \frac{d\varphi}{ds}. \quad (6)$$

Lemma 1. *Let $\gamma = \gamma(t)$ parameterized by t be a nonnull curve and κ be the curvature of γ . Then, we have*

$$\kappa = \frac{\varepsilon(\ddot{\gamma} \cdot \mathcal{J}\dot{\gamma})}{\|\dot{\gamma}\|^3} \quad (7)$$

where $\dot{\gamma} = \frac{d\gamma}{dt}$ and $\varepsilon = 1$ or -1 if γ is timelike or spacelike, respectively.

Proof. If γ is a timelike curve, we have

$$\tanh \varphi = \frac{d\gamma_2}{d\gamma_1} = \frac{\dot{\gamma}_2}{\dot{\gamma}_1}, \quad \varphi = \tanh^{-1} \left(\frac{\dot{\gamma}_2}{\dot{\gamma}_1} \right).$$

Taking a derivative of the angle φ with respect to arc-length parameter s , we get

$$\frac{d\varphi}{ds} = \frac{(\dot{\gamma}_1 \ddot{\gamma}_2 - \ddot{\gamma}_1 \dot{\gamma}_2)}{\dot{\gamma}_1^2 - \dot{\gamma}_2^2} \frac{1}{\sqrt{|-\dot{\gamma}_1^2 + \dot{\gamma}_2^2|}} = \frac{(\dot{\gamma}_1 \ddot{\gamma}_2 - \ddot{\gamma}_1 \dot{\gamma}_2)}{|-\dot{\gamma}_1^2 + \dot{\gamma}_2^2|^{\frac{3}{2}}}. \quad (8)$$

If γ is a spacelike curve, we have

$$\coth \varphi = \frac{\dot{\gamma}_2}{\dot{\gamma}_1}, \quad \varphi = \coth^{-1} \left(\frac{\dot{\gamma}_2}{\dot{\gamma}_1} \right)$$

and from here

$$\frac{d\varphi}{ds} = \frac{(\dot{\gamma}_1 \ddot{\gamma}_2 - \ddot{\gamma}_1 \dot{\gamma}_2)}{\dot{\gamma}_1^2 - \dot{\gamma}_2^2} \frac{1}{\sqrt{|-\dot{\gamma}_1^2 + \dot{\gamma}_2^2|}} = \frac{(\dot{\gamma}_1 \ddot{\gamma}_2 - \ddot{\gamma}_1 \dot{\gamma}_2)}{-|-\dot{\gamma}_1^2 + \dot{\gamma}_2^2|^{\frac{3}{2}}}. \quad (9)$$

Then, we can find the formula (7) by (8) and (9). \square

Lemma 2. i) *Let $f, g : (t_1, t_2) \rightarrow \mathbb{R}$ be differentiable functions with $-f^2 + g^2 = 1$. Fix t_0 with $t_1 < t_0 < t_2$ and suppose θ_0 is such that $f(t_0) = \sinh \theta_0$ and $g(t_0) = \cosh \theta_0$. Then, there exists a unique function $\vartheta : (t_1, t_2) \rightarrow \mathbb{R}$ such that*

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$$\vartheta(t_0) = \theta_0, \quad f(t) = \sinh \vartheta(t), \quad g(t) = \cosh \vartheta(t) \quad (10)$$

for $t_1 < t < t_2$.

- ii) Let $f, g : (t_1, t_2) \rightarrow \mathbb{R}$ be differentiable functions with $-f^2 + g^2 = -1$. Fix t_0 with $t_1 < t_0 < t_2$ and suppose θ_0 is such that $f(t_0) = \cosh \theta_0$ and $g(t_0) = \sinh \theta_0$. Then, there exists a unique function $\vartheta : (t_1, t_2) \rightarrow \mathbb{R}$ such that

$$\vartheta(t_0) = \theta_0, \quad f(t) = \cosh \vartheta(t), \quad g(t) = \sinh \vartheta(t)$$

for $t_1 < t < t_2$.

Proof. i) Let $\mathbf{w} = f\mathbf{i} + g\mathbf{j}$ such that $\mathbf{w}^2 = 1$. If we define

$$\vartheta(t) = \theta_0 + J \int_{t_0}^t \mathbf{w}(u) \mathbf{w}'(u) du,$$

then

$$\frac{d}{dt} (\mathbf{j} \mathbf{w} e^{-J\vartheta}) = 0$$

so that $\mathbf{j} \mathbf{w} e^{-J\vartheta} = c$ for some constant c . Since $\mathbf{w}(t_0) = \mathbf{j} e^{J\theta_0}$, it follows that $c = \pm 1$ and so we get (10). The uniqueness is trivial.

ii) The proof is similar to i). □

Corollary 3. Let γ and β be regular nonnull curves in $\mathcal{M}^{1,1}$ defined on the same interval U and let $t_0 \in U$. Choose θ_0 such that

$$\frac{\gamma'(t_0) \cdot \beta'(t_0)}{\|\gamma'(t_0)\| \|\beta'(t_0)\|} = \cosh \theta_0, \quad \frac{\gamma'(t_0) \cdot \mathcal{J}\beta'(t_0)}{\|\gamma'(t_0)\| \|\beta'(t_0)\|} = \sinh \theta_0$$

or

$$\frac{\gamma'(t_0) \cdot \beta'(t_0)}{\|\gamma'(t_0)\| \|\beta'(t_0)\|} = \sinh \theta_0, \quad \frac{\gamma'(t_0) \cdot \mathcal{J}\beta'(t_0)}{\|\gamma'(t_0)\| \|\beta'(t_0)\|} = \cosh \theta_0.$$

Then there exist a unique differentiable function $\vartheta : I \rightarrow \mathbb{R}$ such that

$$\vartheta(t_0) = \theta_0, \quad \frac{\gamma'(t) \cdot \beta'(t)}{\|\gamma'(t)\| \|\beta'(t)\|} = \cosh \vartheta(t), \quad \frac{\gamma'(t) \cdot \mathcal{J}\beta'(t)}{\|\gamma'(t)\| \|\beta'(t)\|} = \sinh \vartheta(t)$$

or

$$\vartheta(t_0) = \theta_0, \quad \frac{\gamma'(t) \cdot \beta'(t)}{\|\gamma'(t)\| \|\beta'(t)\|} = \sinh \vartheta(t), \quad \frac{\gamma'(t) \cdot \mathcal{J}\beta'(t)}{\|\gamma'(t)\| \|\beta'(t)\|} = \cosh \vartheta(t).$$

In the Lemma 2, we can take $f(t) = -\sinh \vartheta(t)$ and $g(t) = -\cosh \vartheta(t)$ or $f(t) = -\cosh \vartheta(t)$ and $g(t) = -\sinh \vartheta(t)$ if $f(t_0) = -\cosh \theta_0$ and $g(t_0) = -\sinh \theta_0$ or $f(t_0) = -\sinh \theta_0$ and $g(t_0) = -\cosh \theta_0$, respectively. We call ϑ the *hyperbolic angle function* between γ and β determined by θ_0 .

4. Conformal Curves in the de Sitter 2-Space

In this section, we investigate a map Ψ of 2-dimensional de Sitter subspace of $\mathcal{M}^{2,1}$ defined by

$$\mathcal{S}_r^2 = \{\mathbf{a} \in \mathcal{M}^{2,1} : \mathbf{a}^2 = r^2\}$$

onto the extended Minkowski plane $\tilde{\mathcal{M}}^{1,1}$. Let's choose the points $\mathbf{A}_+ = r\mathbf{k}$, $\mathbf{A}_- = -r\mathbf{k}$ and $\mathbf{A}_0 = -r\mathbf{j}$ on \mathcal{S}_r^2 . The generalized stereographic projection $\Gamma : \mathcal{S}_r^2 \setminus \bar{\Lambda} \rightarrow \mathcal{M}^{1,1} \setminus \mathcal{H}_r^1$ is defined by

$$\Gamma(\mathbf{a}) = \mathbf{m} = \frac{2r^2}{\mathbf{a} - r\mathbf{k}} = \frac{ra_1}{r - a_3}\mathbf{i} + \frac{ra_2}{r - a_3}\mathbf{j} - r\mathbf{k} \quad (a_3 \neq r), \quad (11)$$

for $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where

$$\begin{aligned} \bar{\Lambda} &= \{\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathcal{S}_r^2 : x_3 = r\} \text{ and} \\ \mathcal{H}_r^1 &= \{x\mathbf{i} + y\mathbf{j} \in \mathcal{M}^{1,1} : -x^2 + y^2 = -r^2\}. \end{aligned}$$

Also, the map Γ is one to one, onto and a conformal map (see [12]). So, we can extend the map Γ to extended Minkowski plane with the map $\sigma : \mathcal{S}_r^2 \rightarrow \tilde{\mathcal{M}}^{1,1} \setminus \mathcal{H}_r^1$ given by

$$\begin{cases} \sigma(\mathbf{a}) = \mathbf{m} & \text{for } \mathbf{a} \in \mathcal{S}_r^2 \setminus \bar{\Lambda} \\ \sigma(\bar{\Lambda}) = I_\infty \end{cases} \quad (12)$$

such that $\sigma(p\mathbf{i} + p\mathbf{j} + r\mathbf{k}) = (p\mathbf{i} + p\mathbf{j})^{-1}$ and $\sigma(p\mathbf{i} - p\mathbf{j} + r\mathbf{k}) = (p\mathbf{i} - p\mathbf{j})^{-1}$ for all $p \in \mathbb{R} \cup \{\infty\}$. The inverse generalized stereographic projection $\sigma^{-1} : \tilde{\mathcal{M}}^{1,1} \setminus \mathcal{H}_r^1 \rightarrow \mathcal{S}_r^2$ can be represented by

$$\begin{aligned} \sigma^{-1}(\mathbf{m}) &= \mathbf{a} = \frac{2r^2\mathbf{m} + r\mathbf{m}^2\mathbf{k}}{\mathbf{m}^2} = -\frac{r\mathbf{m}\mathbf{k}\mathbf{m}}{\mathbf{m}^2} \\ &= \frac{2xr^2}{\mathbf{m}^2}\mathbf{i} + \frac{2yr^2}{\mathbf{m}^2}\mathbf{j} + \left(\frac{-2r^3 + r\mathbf{m}^2}{\mathbf{m}^2}\right)\mathbf{k}, \\ \sigma^{-1}(I_\infty) &= \bar{\Lambda} \end{aligned}$$

for $\mathbf{m} = x\mathbf{i} + y\mathbf{j} - r\mathbf{k}$ from (12).

Let be $N = \{\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \in \mathcal{S}_r^2 : a_2 = -r\}$ and choose the linear-fractional transformation $T_{\mathbf{u}} : \tilde{\mathcal{M}}^{1,1} \rightarrow \tilde{\mathcal{M}}^{1,1}$ defined by

$$T_{\mathbf{u}}(\mathbf{w}) = (-i\mathbf{u}\mathbf{w} + r\mathbf{u})(\mathbf{w} + r\mathbf{j})^{-1}\mathbf{i} \quad (13)$$

where $\mathbf{u} = r\mathbf{i} + r\mathbf{j}$ is a null vector. Then, we can establish a map $\Psi = T_{\mathbf{u}} \circ \sigma : \mathcal{S}_r^2 \rightarrow \tilde{\mathcal{M}}^{1,1} \setminus \mathcal{H}_r^1$. The image of N under Ψ is in I_∞ . The transformation Ψ is a bijective conformal map and maps \mathbf{A}_0 to $(0\mathbf{i} + 0\mathbf{j})^{-1}$, \mathbf{A}_+ to $\tilde{\mathbf{u}} = r\mathbf{i} - r\mathbf{j}$ and \mathbf{A}_- to \mathbf{u} . The explicit expression of the map Ψ can be given by

$$\begin{aligned} \Psi(\mathbf{a}) &= \mathbf{n} = \left(\frac{-2r\mathbf{i}\mathbf{u}}{\mathbf{a} - r\mathbf{k}} - \mathbf{i}\mathbf{u}\mathbf{k} + \mathbf{u}\right) \left(\frac{2r}{\mathbf{a} - r\mathbf{k}} + \mathbf{k} + \mathbf{j}\right)^{-1} \mathbf{i} \\ &= \frac{r(-a_1 + a_2 + r)}{a_2 + r}\mathbf{i} - \frac{ra_3}{a_2 + r}\mathbf{j} \quad (a_2 \neq -r), \end{aligned}$$

for $\mathbf{a} \in \mathcal{S}_r^2 \setminus N$ and $\Psi(N) = I_\infty$.

The inverse mapping $\Psi^{-1} : \tilde{\mathcal{M}}^{1,1} \setminus \mathcal{H}_r^1 \rightarrow \mathcal{S}_r^2$ can be given as the following

$$\begin{aligned} \Psi^{-1}(\mathbf{n}) &= \mathbf{a} = 2r(\mathbf{i}\mathbf{u} - \mathbf{n}\mathbf{i})(-\mathbf{i}\mathbf{u}\mathbf{k} + \mathbf{u} + \mathbf{n}(\mathbf{i}\mathbf{k} + \mathbf{j}))^{-1} + r\mathbf{k} \\ &= \frac{2r^2(r-x)}{2rx-x^2+y^2}\mathbf{i} + \frac{r(2r^2+x^2-2rx-y^2)}{2rx-x^2+y^2}\mathbf{j} + \frac{-2r^2y}{2rx-x^2+y^2}\mathbf{k} \end{aligned}$$

for $\mathbf{n} = x\mathbf{i} + y\mathbf{j}$ and $\Psi^{-1}(I_\infty) = N$ by using $\Psi^{-1} = \sigma^{-1} \circ T_{\mathbf{u}}^{-1}$.

We can see that the map Ψ transforms the timelike pseudo-circle \mathcal{P}_0 on \mathcal{S}_r^2 given by $\mathbf{v}^2 = r^2$ for $\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} \in \mathcal{S}_r^2$ to the real axis of $\mathcal{M}^{1,1}$ using (3), (11) and (13). Let Ω be a one-parameter family of the pseudo-circles \mathcal{P}_t on \mathcal{S}_r^2 tangent to \mathcal{P}_0 at \mathbf{A}_0 such that the equations of the image of \mathcal{P}_t under the generalized stereographic projection σ in $\tilde{\mathcal{M}}^{1,1}$ are given by

$$(\mathbf{v} + t\mathbf{j})^2 = (r-t)^2, \quad t \in \mathbb{R}.$$

The one-parameter family Ω is mapped onto a bunch of the horizontal lines under $T_{\mathbf{u}}$ using (2) and (3) in $\mathcal{M}^{1,1}$.

Let $\beta : I \rightarrow \mathcal{S}_r^2$ be a non-lightlike curve defined on an open interval $I \subset \mathbb{R}$. So, $\alpha = \Psi \circ \beta : I \rightarrow \tilde{\mathcal{M}}^{1,1} \setminus \mathcal{H}_r^1$ is a non-lightlike curve in the extended Minkowski plane. We denote the group of the conformal transformations of the de Sitter 2-space as $\mathbf{Conf}(\mathcal{S}_r^2)$.

Lemma 4. *Let $\beta_i : I \rightarrow \mathcal{S}_r^2$, $i = 1, 2$ be two non-lightlike curves and $\alpha_i = \Psi \circ \beta_i$ be corresponding curves in $\tilde{\mathcal{M}}^{1,1} \setminus \mathcal{H}_r^1$. Then if $f_\Psi : \mathcal{S}_r^2 \rightarrow \mathcal{S}_r^2$ is a bijection conformal map on \mathcal{S}_r^2 and $f_\Psi(\beta_1) = \beta_2$, then $f = \Psi \circ f_\Psi \circ \Psi^{-1}$ is a conformal map satisfies the equality $f(\alpha_1) = \alpha_2$. Furthermore, f is a similarity if $f_\Psi(N) = N$.*

Proof. Since Ψ , f_Ψ and Ψ^{-1} are conformal, the transformation f is also a conformal map and it can be written as

$$f(\alpha_1) = \Psi \circ f_\Psi \circ \Psi^{-1}(\alpha_1) = \Psi(\beta_2) = \alpha_2.$$

Also, we can say that if a conformal transformation maps I_∞ to I_∞ in the extended double plane, it is a similarity (see [4] for Euclidean plane). Therefore, f is a similarity if we have $f_\Psi(N) = N$. \square

Let G be a set of the transformations $f_\Psi \in \mathbf{Conf}(\mathcal{S}_r^2)$ preserving a fixed point $\mathbf{Q} \in \mathcal{S}_r^2$. G is a subgroup of $\mathbf{Conf}(\mathcal{S}_r^2)$. Moreover, we have a one-parameter family Ω_d of pseudo-circles on \mathcal{S}_r^2 with the same tangent line d , where $d \subset \mathbb{L}_{\mathbf{Q}}(\mathcal{S}_r^2)$ is a fixed tangent line passing through \mathbf{Q} .

Theorem 5. *Suppose that $\beta_i : I \rightarrow \mathcal{S}_r^2$ are two non-lightlike curves, which have the same causal characters, of class C^2 defined on an open interval $I \subset \mathbb{R}$, ($i = 1, 2$) and there exist a finite subset $\emptyset \subseteq \mathcal{T} = \{t_1, \dots, t_k\}$ of I satisfying the following conditions:*

- 1) $\beta_i(t) \neq \mathbf{Q}$ for $t \in I \setminus \mathcal{T}$
- 2) $\beta_i(t) = \mathbf{Q}$ for $t \in \mathcal{T}$.

Let $\phi_i(t) = \angle(\beta_i(t), \mathcal{P}(t))$, $t \in I \setminus \mathcal{T}$, be the Lorentzian angle at the point $\beta_i(t)$ between β_i and the unique pseudo-circle $\mathbf{C} \in \Omega_d$ passing through $\beta_i(t)$, and $\hat{\phi}_m = \angle(\beta_1(t_m), \beta_2(t_m))$ be the Lorentzian angle between β_1 and

β_2 at the point $\beta_1(t_m) = \beta_2(t_m) = \mathbf{Q}$ for $m = 1, \dots, k$. Then, we have $f_\Psi \in G$ satisfying $f_\Psi(\beta_1) = \beta_2$ if and only if there is a constant ϕ_0 such that ϕ_0 satisfies the following conditions:

$$\begin{aligned} i) \quad & \phi_1(t) = \pm \phi_2(t) + \phi_0 \quad \text{for any } t \in I \setminus \mathcal{T} \\ ii) \quad & \tilde{\phi}_m = \phi_0 \quad \text{for } m = 1, \dots, k. \end{aligned}$$

Proof. We can say that there is an orientation-preserving isometry \mathcal{R} of \mathcal{S}_r^2 satisfying $\mathcal{R}(\mathbf{Q}) = \mathbf{A}_0$ and $\Omega_d \xrightarrow{\mathcal{R}} \Omega$ such that the conditions $f_\Psi \in \mathbf{Conf}(\mathcal{S}_r^2)$ and $f_\Psi(\mathbf{Q}) = \mathbf{Q}$ are equivalent to the conditions $\mathcal{R}^{-1} \circ f_\Psi \circ \mathcal{R} \in \mathbf{Conf}(\mathcal{S}_r^2)$ and $(\mathcal{R}^{-1} \circ f_\Psi \circ \mathcal{R})(\mathbf{Q}) = \mathbf{Q}$. Then, we may assume that $\mathbf{Q} = \mathbf{A}_0$ and $\Omega_d = \Omega$ without loss of generality.

As we know that Ψ is a conformal map and $\Psi(\mathcal{P})$ is a horizontal line, we can write

$$\phi_i = \angle(\beta_i, \mathcal{P}) = \angle(\alpha_i, \Psi(\mathcal{P})) = \arg\left(\frac{d\alpha_i/dt}{\|d\alpha_i/dt\|}\right)$$

in \mathbb{P} .

Firstly we consider that $f_\Psi(\beta_1) = \beta_2$ for $f_\Psi \in G$. We have that $f = \Psi \circ f_\Psi \circ \Psi^{-1}$ is a similarity transformation and $f(\alpha_1) = \alpha_2$. Therefore, we get

$$\frac{d\alpha_1/dt}{\|d\alpha_1/dt\|} = \mathbf{B} \frac{d\alpha_2/dt}{\|d\alpha_2/dt\|}$$

for some fixed spinor \mathbf{B} . Then, $\arg(d\alpha_1/dt) = \pm \arg(d\alpha_2/dt) + \phi_0$ or $\phi_1(t) = \pm \phi_2(t) + \phi_0$, where $\mathbf{B} = e^{\phi_0 J}$. From here, ϕ_0 is the angle of the hyperbolic rotation which is a component of f and $\tilde{\phi}_m = \phi_0$ for $m = 1, \dots, k$.

Now, assume that $\phi_1(t) = \pm \phi_2(t) + \phi_0$, $\phi_0 = \text{const.}$ for $t \in I \setminus \mathcal{T}$ and $\tilde{\phi}_m = \phi_0$ for $m = 1, \dots, k$. We consider α_i , $i = 1, 2$, as smooth regular curves. From (6) we can write

$$\frac{d\phi_1(t)}{dt} = \pm \frac{d\phi_2(t)}{dt} \quad \text{or} \quad \left\| \frac{d\alpha_2}{dt} \right\| = \left(\frac{\kappa_1}{\kappa_2} \right) \left\| \frac{d\alpha_1}{dt} \right\|, \quad (14)$$

where κ_i is the oriented curvature of α_i . So, we can say that there is a transformation $g \in \mathbf{Conf}(\tilde{\mathcal{M}}^{1,1} \setminus \mathcal{H}_r^1)$ such that $g(\alpha_1) = \alpha_2$. However, any conformal transformation of double plane is either a composition of a Lorentzian motion and an inversion or a similarity. Since the fact that g is not a similarity give rise to a contradiction with the Eq. (14), we get $f_\Psi = \Psi^{-1} \circ g \circ \Psi \in G$ and $f_\Psi(\beta_1) = \beta_2$. It is obvious when α_1 and α_2 are straight lines. \square

5. De Sitter Loxodromes

In the Euclidean plane, the unique plane curves with the constant similarity invariant $\tilde{\kappa} \neq 0$ are logarithmic spirals defined by

$$\varsigma(t) = a(e^{bt} \cos t, e^{bt} \sin t)$$

so that they are the self-similar curves [7]. The tangent-radius angle of a logarithmic spiral is a constant. Moreover, a spherical loxodrome is the pre-image

under a stereographic projection of a logarithmic spiral in the Euclidean 3-space [8]. In this section, we shall describe the pseudo-spherical loxodromes on the de Sitter-2 Space.

The curves parameterized by

$$\varsigma_1(t) = (ae^{bt} \cosh t)\mathbf{i} + (ae^{bt} \sinh t)\mathbf{j} \quad \text{or} \quad \varsigma_2(t) = (ae^{bt} \sinh t)\mathbf{i} + (ae^{bt} \cosh t)\mathbf{j} \quad (15)$$

are self-similar curves with the constant similarity invariant $\tilde{\kappa} \neq 0$ in the Minkowski plane [16]. Therefore, we can say that the non-lightlike curves ς_1 and ς_2 are the hyperbolic logarithmic spirals of Minkowski plane.

Let $\gamma : I \rightarrow \mathcal{M}^{1,1}$ be a nonnull curve which does not pass through the origin. There exists a unique differentiable function $\tau : I \rightarrow \mathbb{R}$ from Lemma 2 such that

$$\frac{\gamma'(t) \cdot \gamma(t)}{\|\gamma'(t)\| \|\gamma(t)\|} = \pm \cosh \tau(t), \quad \frac{\gamma'(t) \cdot \mathcal{J}\gamma(t)}{\|\gamma'(t)\| \|\gamma(t)\|} = \pm \sinh \tau(t) \quad (16)$$

or

$$\frac{\gamma'(t) \cdot \gamma(t)}{\|\gamma'(t)\| \|\gamma(t)\|} = \pm \sinh \tau(t), \quad \frac{\gamma'(t) \cdot \mathcal{J}\gamma(t)}{\|\gamma'(t)\| \|\gamma(t)\|} = \pm \cosh \tau(t) \quad (17)$$

for $t \in I$. $\tau(t)$ presents the hyperbolic angle between the radius vector $\gamma(t)$ and the tangent vector $\gamma'(t)$. It is called $\tau(t)$ the *hyperbolic tangent-radius angle* of γ .

Lemma 6. $\gamma : I \rightarrow \mathcal{M}^{1,1}$ be a nonnull curve which does not pass through the origin. The following conditions are equivalent:

- i) The hyperbolic tangent-radius angle τ is constant;
- ii) γ is a reparametrization of an hyperbolic logarithmic spiral.

Proof. Let's γ be a timelike curve. We can write $\gamma(t) = a\mathbf{ie}^{J\theta}$ and $\gamma'(t) = (a'\mathbf{i} + a\theta'\mathbf{j})e^{J\theta}$ so that

$$\|\gamma(t)\| = a, \quad \|\gamma'(t)\| = \sqrt{-a'^2 + a^2\theta'^2}$$

Suppose that i) holds and let δ be constant value of $\tau(t)$. If there exists the Eq. (16) for γ , then using the last equation of (5), we have

$$\frac{\gamma' \cdot \gamma}{\|\gamma'(t)\| \|\gamma(t)\|} = \frac{-a'}{\sqrt{-a'^2 + a^2\theta'^2}} - \frac{a\theta'}{\sqrt{-a'^2 + a^2\theta'^2}} \mathbf{ij}$$

or

$$\cosh \delta = \frac{-a'}{\sqrt{-a'^2 + a^2\theta'^2}} \quad \text{and} \quad \sinh \delta = \frac{-a\theta'}{\sqrt{-a'^2 + a^2\theta'^2}}$$

so that

$$\frac{a'}{a} = \theta' \coth \delta.$$

The solution of this differential equation is

$$a = ce^{(\coth \delta)\theta}$$

where c is a constant. From here, we can obtain

$$\gamma(t) = c\mathbf{ie}^{(\coth \delta + J)\theta(t)}.$$

which implies that γ is a reparametrization of the hyperbolic logarithmic spiral. We can similarly follow the same operations for a spacelike curve.

One can easily find the hyperbolic logarithmic spiral defined by (15) has a constant hyperbolic tangent-radius angle. \square

A meridian on a de Sitter 2-space is branches of an hyperbola which is obtained by the intersection of a plane contains Z -axis with \mathcal{S}_r^2 . A *de Sitter loxodrome* or *de Sitter rhumb line* is a curve on \mathcal{S}_r^2 which meets each meridian of the de Sitter 2-space at the same angle. Then, we use the generalized stereographic projection in order to find the parametrization of a de Sitter loxodrome.

Any pseudo-circle or line given by (1) in the extended Minkowski plane can be given implicitly by an equation of the form

$$a(-x^2 + y^2) + bx + cy + d = 0 \quad (18)$$

where a, b, c, d are real constants. The Eq. (18) under σ^{-1} is mapped into

$$bX + cY + (ar - d/r)Z + ar^2 + d = 0, \quad (19)$$

which is the equation of a plane in $\mathcal{M}^{2,1}$. This plane meets the de Sitter 2-space \mathcal{S}_r^2 in a meridian. In case of $a = d = 0$ in the Eq. (18), we get a straight line passes through the origin. From (19), the plane containing the image curve also include the Z -axis. Thus, the image of a straight line passes through the origin is a meridian on \mathcal{S}_r^2 .

Lemma 7. *A de Sitter loxodrome is the image of an hyperbolic logarithmic spiral under the inverse generalized stereographic projection.*

Proof. Lemma 6 implies that an hyperbolic logarithmic spiral meets every line passes through the origin at the same hyperbolic angle. The inverse generalized stereographic projection transforms each of these lines into a meridian of the de Sitter 2-space. Since σ^{-1} is a conformal map, it maps each hyperbolic logarithmic spiral onto a de Sitter loxodrome. \square

Using the Lemma 7, the parametrizations of de Sitter loxodromes are given by

$$\begin{aligned} dlox_1(t) = \sigma^{-1}(\varsigma_1) = & \frac{1}{r^2 - a^2 e^{2bt}} ((2ar^2 e^{bt} \cosh t)\mathbf{i} + (2ar^2 e^{bt} \sinh t)\mathbf{j} \\ & + r(-a^2 e^{2bt} - r^2)\mathbf{k}) \end{aligned}$$

and

$$\begin{aligned} dlox_2(t) = \sigma^{-1}(\varsigma_2) = & \frac{1}{r^2 + a^2 e^{2bt}} ((2ar^2 e^{bt} \sinh t)\mathbf{i} + (2ar^2 e^{bt} \cosh t)\mathbf{j} \\ & + r(a^2 e^{2bt} - r^2)\mathbf{k}). \end{aligned}$$

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H. Simsek and M. Özdemir

Hakan Simsek and Mustafa Özdemir
Department of Mathematics
Akdeniz University
Antalya
Turkey
e-mail: hakansimsek@akdeniz.edu.tr

Mustafa Özdemir
e-mail: mozdemir@akdeniz.edu.tr

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