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On the Eigenvalues and Eigenvectors of a Lorentzian Rotation Matrix by Using Split Quaternions

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Abstract. In this paper, we examine eigenvalue problem of a rotation matrix in Minkowski 3 space by using split quaternions. We express the eigenvalues and the eigenvectors of a rotation matrix in term of the coefficients of the corresponding unit timelike split quaternion. We give the characterizations of eigenvalues (complex or real) of a rotation matrix in Minkowski 3 space according to only first component of the corresponding quaternion. Moreover, we find that the casual characters of rotation axis depend only on first component of the corresponding quaternion. Finally, we give the way to generate an orthogonal basis for $\mathbb{E}_1^3$ by using eigenvectors of a rotation matrix.

Keywords. Quaternions, Split Quaternions, Rotation Matrix.

1. Introduction

Irish Mathematician Sir William Rowan Hamilton first described the quaternions in 1843. This description is a kind of extension complex numbers to higher spatial dimensions. So the set of quaternions can be represented as

$$\mathbb{H} = \{q = q_1 + q_2 i + q_3 j + q_4 k; \quad q_1, q_2, q_3, q_4 \in \mathbb{R}\}$$

where

$$i^2 = j^2 = k^2 = -1 \text{ and } ijk = -1.$$

The set of quaternions is a member of noncommutative division algebra, [2].

In 1849, James Cockle introduced coquaternions, which can be represented as

$$\widehat{\mathbb{H}} = \{q = q_1 + q_2 i + q_3 j + q_4 k; \quad q_1, q_2, q_3, q_4 \in \mathbb{R}\}.$$

Here, the imaginary units satisfy the relations

$$i^2 = -1, j^2 = k^2 = 1 \text{ and } ijk = 1.$$

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Due to the division of imaginary units into positive and negative terms, the coquaternions came to be called split quaternions. The set of split quaternions is noncommutative, too. Contrary to quaternion algebra, the set of split quaternions contains zero divisors, nilpotent elements and nontrivial idempotents, [1], [3], [4], [5].

This paper is concerned with the eigenvalue problem of a rotation matrix in Minkowski 3 space. We examine the eigenvalues and eigenvector of a 3×3 rotation matrix by using unit timelike split quaternions for Minkowski 3 space. We express the eigenvalues and the eigenvectors of a rotation matrix in term of the coefficients of the corresponding quaternion. Also, we give the characterizations of eigenvalues (complex or real) of a rotation matrix in Minkowski 3 space according to only first component of the corresponding unit timelike quaternion. Moreover, we find that the casual character of rotation axis depends only on first component of the corresponding unit timelike quaternion. As a conclusion, we give a way to generate an orthogonal basis for $\mathbb{E}_1^3$ by using eigenvectors of a rotation matrix.

2. Split Quaternions and Rotations in $\mathbb{E}_1^3$

The set of split quaternions can be represented as

$$\widehat{\mathbb{H}} = \{q = q_1 + q_2 i + q_3 j + q_4 k; \quad q_1, q_2, q_3, q_4 \in \mathbb{R}\}$$

where $i^2 = -1$, $j^2 = k^2 = 1$ and $ijk = 1$.

We write any split quaternion in the form $q = (q_1, q_2, q_3, q_4) = S_q + \overrightarrow{V_q}$ where $S_q = q_1$ denotes the scalar part of q and $\overrightarrow{V_q} = q_2 i + q_3 j + q_4 k$ denotes vector part of q. If $S_q = 0$ then q is called pure split quaternion and the set of pure quaternions can be identified with Minkowski 3 space. Here, the Minkowski 3 space is Euclidean space with Lorentzian inner product

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_L = -u_1 v_1 + u_2 v_2 + u_3 v_3$$

where $\overrightarrow{u} = (u_1, u_2, u_3)$, $\overrightarrow{v} = (v_1, v_2, v_3) \in \mathbb{E}^3$ and denoted by $\mathbb{E}_1^3$. And the rotations in Minkowski 3 space can be stated with split quaternions such as expressing the Euclidean rotations using quaternions, [4].

The conjugate of a split quaternion $q = q_1 + q_2 i + q_3 j + q_4 k \in \widehat{\mathbb{H}}$ is denoted by $\overline{q}$ and it is

$$\overline{q} = S_q - \overrightarrow{V_q} = q_1 - q_2 i - q_3 j - q_4 k.$$

For $p, q \in \widehat{\mathbb{H}}$, the sum and product of split quaternions p and q are

$$\begin{aligned} p + q &= S_p + S_q + \overrightarrow{V_p} + \overrightarrow{V_q}, \\ pq &= S_p S_q + \left\langle \vec{V}p, \vec{V}q \right\rangle_L + S_p \vec{V}q + S_q \vec{V}p + \vec{V}p \wedge_L \vec{V}q, \end{aligned}$$

respectively. Here $\langle\ ,\ \rangle_L$ and $\wedge_L$ denote Lorentzian inner and vector product and are defined as

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_L = -u_1v_1 + u_2v_2 + u_3v_3,$$

$$\overrightarrow{u} \wedge_L \overrightarrow{v} = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

for vectors $\overrightarrow{u} = (u_1, u_2, u_3)$ and $\overrightarrow{v} = (v_1, v_2, v_3)$ of Minkowski 3 space, respectively. The norm of split quaternion q is defined by

$$N_q = \sqrt{|q\overline{q}|} = \sqrt{|q_1^2 + q_2^2 - q_3^2 - q_4^2|}.$$

If $N_q = 1$ then q is called unit split quaternion and $q_0 = q/N_q$ is a unit split quaternion for $N_q \neq 0$. The product

$$I_q = q\overline{q} = \overline{q}q = q_1^2 + q_2^2 - q_3^2 - q_4^2$$

determines the character of a split quaternion. A split quaternion is spacelike, timelike or lightlike (null) if $I_q < 0$, $I_q > 0$ or $I_q = 0$, respectively.
For further information, see [4] and [5].
The set of timelike split quaternions, which is denoted by

$$\mathbb{T}\widehat{\mathbb{H}} = \{q = (q_1, q_2, q_3, q_4) : q_1, q_2, q_3, q_4 \in \mathbb{R},\ I_q > 0\},$$

forms a group under the split quaternion product. Any timelike split quaternion can be represented in polar form similar to quaternions as follows:

i) Every timelike split quaternion with spacelike vector part can be written in the form

$$q = N_q (\cosh\theta + \vec{\varepsilon}_0 \sinh\theta)$$

where $\cosh\theta = \frac{q_1}{N_q}$, $\sinh\theta = \frac{\sqrt{-q_2^2+q_3^2+q_4^2}}{N_q}$, $\vec{\varepsilon}_0 = \frac{q_2 i + q_3 j + q_4 k}{\sqrt{-q_2^2+q_3^2+q_4^2}}$ is a spacelike unit vector in $\mathbb{E}_1^3$ and $\vec{\varepsilon}_0 * \vec{\varepsilon}_0 = 1$,

ii) Every timelike split quaternion with timelike vector part can be written in the form

$$q = N_q (\cos\theta + \vec{\varepsilon}_0 \sin\theta)$$

where $\cos\theta = \frac{q_1}{N_q}$, $\sin\theta = \frac{\sqrt{q_2^2-q_3^2-q_4^2}}{N_q}$, $\vec{\varepsilon}_0 = \frac{q_2 i + q_3 j + q_4 k}{\sqrt{q_2^2-q_3^2-q_4^2}}$ is a timelike unit vector in $\mathbb{E}_1^3$ and $\vec{\varepsilon}_0 * \vec{\varepsilon}_0 = -1$.

The set of unit timelike split quaternions is denoted by

$$\mathbb{T}\widehat{\mathbb{H}}_1 = \{q = (q_1, q_2, q_3, q_4) : q_1, q_2, q_3, q_4 \in \mathbb{R},\ I_q > 0,\ N_q = 1\}.$$

A matrix is called pseudo orthogonal if it preserves the length of vectors in the Minkowski 3 space. That is, if $\langle A\overrightarrow{u}, A\overrightarrow{u} \rangle_L = \langle \overrightarrow{u}, \overrightarrow{u} \rangle_L$ for all $\overrightarrow{u} \in \mathbb{E}_1^3$. Columns (or rows) of the a pseudo orthogonal matrix form an orthonormal basis of $\mathbb{E}_1^3$. Pseudo orthogonal matrices are also characterized by their inverses. A matrix A is an pseudo orthogonal if and only if $I^* A^T I^* = A^{-1}$ where A^T is transpose of the matrix A and

$$I^* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, A is a pseudo orthogonal matrix if and only if $I^* A^T I^* A = I$. If we take the determinant of both sides of this equation, we see that $\det(A) = \pm 1$. Any pseudo orthogonal matrix with determinant 1 is a **Lorentzian rotation matrix**. The set of the Lorentzian rotation matrices of $\mathbb{E}_1^3$ can be expressed as;

$$SO(1,2) = \{R \in M_3(\mathbb{R}) : R^T I^* R = I^*, \ \det R = 1\}.$$

For any two vectors $\overrightarrow{u}$, $\overrightarrow{v} \in \mathbb{E}_1^3$, the matrix product notation $u^T I^* v$ and the Lorentzian inner product notation $\langle \overrightarrow{u}, \overrightarrow{v} \rangle_L$ can be interchanged. Therefore, we may write the equality

$$\langle A\overrightarrow{u}, A\overrightarrow{v} \rangle_L = (A\overrightarrow{u})^t I^* A\overrightarrow{v} = u^T A^T I^* A v = u^T I^* v.$$

for any $\overrightarrow{u}$, $\overrightarrow{v} \in \mathbb{E}_1^3$. So, if we use the equality $I^* A^T I^* A = I$, we have

$$\langle A\overrightarrow{u}, A\overrightarrow{v} \rangle_L = u^T I^* v = \langle \overrightarrow{u}, \overrightarrow{v} \rangle_L .$$

That is, the rotation matrices in the Minkowski 3 space preserve Lorentzian inner product, angles and lenghts. Also, kind of rotation angle (spherical or hyperbolic) and rotation axis change with respect to rotation matrix, [4].
We can generate a Lorentzian rotation matrix by a unit timelike split quaternion as follows:

$$R(q_1, q_2, q_3, q_4) = \begin{bmatrix} q_1^2+q_2^2+q_3^2+q_4^2 & 2q_1q_4-2q_2q_3 & -2q_1q_3-2q_2q_4 \\ 2q_2q_3+2q_4q_1 & q_1^2-q_2^2-q_3^2+q_4^2 & -2q_3q_4-2q_2q_1 \\ 2q_2q_4-2q_3q_1 & 2q_2q_1-2q_3q_4 & q_1^2-q_2^2+q_3^2-q_4^2 \end{bmatrix}. \tag{1}$$

For a given Lorentzian rotation matrix in $\mathbb{E}_1^3$, we can find a unit timelike split quaternion corresponding to rotation matrix using the following formulas;

$$\begin{aligned} q_1^2 &= \frac{1}{4}(1 + R_{11} + R_{22} + R_{33}), \\ q_2 &= \frac{1}{4q_1}(R_{32} - R_{23}), \\ q_3 &= -\frac{1}{4q_1}(R_{13} + R_{31}), \\ q_4 &= \frac{1}{4q_1}(R_{21} + R_{12}), \end{aligned}$$

for $q_1 \neq 0$. When, $q_1 = 0$, we can find corresponding unit timelike quaternion using the equations

$$q_3 = -\frac{1}{2q_2} R_{12},$$
$$q_4 = -\frac{1}{2q_2} R_{13},$$
$$q_2^2 = 1 + q_3^2 + q_4^2.$$

It is enough to determine the timelike quaternion since $0 < q_1^2 + q_2^2 - q_3^2 - q_4^2$. When $q_1 = 0$, we get $0 < q_2^2 - q_3^2 - q_4^2$ or $q_2 \neq 0$.
The function $\varphi : S_2^3 \simeq \mathbb{T}\widehat{\mathbb{H}}_1 \rightarrow SO(1,2)$, which sends $q = (q_1, q_2, q_3, q_4)$ to the matrix R in equation 1, is a homomorphism of groups. The kernel of φ is $\{\pm 1\}$, so that the rotation matrix corresponds to the pair $\pm q$ of the unit timelike quaternion. In particular, $SO(1,2)$ is isomorphic to the quotient group $\mathbb{T}\widehat{\mathbb{H}}_1 \diagup \{\pm 1\}$ from the first isomorphism theorem. In another words, for every rotation in the Minkowski 3 space $\mathbb{E}_1^3$, there are two unit timelike quaternions that determine this rotation. These timelike quaternions are q and $-q$. So, for every rotation matrix, we can find only one unit timelike split quaternion whose first component is positive, [4].
Also, the type of rotation is expressed by timelike quaternions with following theorems;

Theorem 1. [4] *Let $q = \cosh\theta + \vec{\varepsilon}_0 \sinh\theta$ be a unit timelike quaternion with spacelike vector part and $\vec{\varepsilon}_0$ be a Lorentzian vector. Then the transformation $R(q)$ is a rotation through hyperbolic angle 2θ about the spacelike axis $\vec{\varepsilon}_0$.*

Theorem 2. [4] *Let $q = \cos\theta + \vec{\varepsilon}_0 \sin\theta$ be a unit timelike quaternion with timelike vector part and $\vec{\varepsilon}_0$ be a Lorentzian vector. Then the transformation $R(q)$ is a rotation through 2θ about the timelike axis $\vec{\varepsilon}_0$.*

3. Eigenvalues and Eigenvectors of A Rotation Matrix in $\mathbb{E}_1^3$

In this part, we examine the eigenvalues and eigenvectors of a Lorentzian rotation matrix in $\mathbb{E}_1^3$. Let eigenvalues of the Lorentzian rotation matrix A be λ_1, λ_2 and λ_3. So, characteristic polynomial of the A is

$$\triangle_A(x) = \det(xI - A) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3).$$

If we write $x = 0$, we find $\det A = \lambda_1 \lambda_2 \lambda_3 = 1$. It means that the product of the eigenvalues of a rotation matrix is the determinant of this matrix.

Theorem 3. *One of the eigenvalues of a 3×3 rotation matrix in Minkowski 3 space is 1 and the corresponding eigenvector is the rotation axis.*

Proof. Let A be a Lorentzian rotation matrix in $\mathbb{E}_1^3$. Characteristic polynomial of the matrix A is $\Delta_A(x) = \det(xI - A)$. We can write

$$\det(I - A) = \det A \det(I - A),$$

since $\det A = 1$. Using the properties of transpose and determinant, we obtain

$$\begin{aligned}\det(I - A) &= \det(I^* A^T I^*)\det(I - A)\\ &= \det(I^* A^T I^* - I)\\ &= \det(I^* A I^* - I)^T\\ &= \det(A - I).\end{aligned}$$

Besides, since A is a 3×3 matrix, we have $\det(A - I) = -\det(I - A)$. So,

$$\det(A - I) = -\det(A - I).$$

Thus, $\det(A - I) = 0$ which means one of the roots of characteristic equation of A is 1. Using the matrix given in equation 1, one can find the eigenvector of R corresponding to the eigenvalue $\lambda_1 = 1$ as

$$\overrightarrow{c_1}(q_1, q_2, q_3, q_4) = \begin{bmatrix} q_2 \\ q_3 \\ q_4 \end{bmatrix}.$$

On the other hand, this vector is the rotation axis of R by theorems 1 and 2. Thus, we have proved the theorem. □

Theorem 4. *Let A be a Lorentzian rotation matrix in $\mathbb{E}_1^3$. Then the followings are satisfied;*

i) *If the rotation axis of A is timelike vector, then eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = e^{i\theta} = \cos\theta + i\sin\theta$ and $\lambda_3 = e^{-i\theta} = \cos\theta - i\sin\theta$.*

ii) *If the rotation axis of A is spacelike vector, then eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = e^{\theta} = \cosh\theta + \sinh\theta$ and $\lambda_3 = e^{-\theta} = \cosh\theta - \sinh\theta$.*

Proof. Let A be a Lorentzian rotation matrix in $\mathbb{E}_1^3$. We will examine the eigenvector of A is timelike or spacelike, separately.

i) Let $\overrightarrow{u_1}$ be a unit timelike vector such that $A\overrightarrow{u_1} = \overrightarrow{u_1}$. Thus, unit timelike vector $\overrightarrow{u_1}$ is the rotation axis of A by theorem 3. Take a unit spacelike vector $\overrightarrow{u_2}$ perpendicular to $\overrightarrow{u_1}$, then $\overrightarrow{u_3} = \overrightarrow{u_1} \wedge_L \overrightarrow{u_2}$ will be a unit spacelike vector. Then $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$ is a right handed basis for $\mathbb{E}_1^3$. Let $\{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ be standard orthogonal basis of $\mathbb{E}_1^3$ and B be a matrix such that $B\overrightarrow{e_i} = \overrightarrow{u_i}$. Then, B is a pseudo orthogonal matrix because its columns are orthonormal basis for $\mathbb{E}_1^3$. We compute the matrix $B^{-1}AB$. We know that the product of two pseudo orthogonal matrices is pseudo orthogonal and inverse of a pseudo orthogonal matrix is pseudo orthogonal. So, $B^{-1}AB$ is pseudo orthogonal. Moreover

$$B^{-1}AB\overrightarrow{e_1} = B^{-1}A\overrightarrow{u_1} = B^{-1}\overrightarrow{u_1} = \overrightarrow{e_1}.$$

Therefore, the first column of $B^{-1}AB$ is $\overrightarrow{e_1}$. The first column must be orthogonal to other two columns. So, $B^{-1}AB$ will be in the following form;

$$B^{-1}AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{bmatrix}.$$

Also, second and third column of $B^{-1}AB$ are orthogonal to each other. That is, $ac + bd = 0$. On the other hand, determinant of $B^{-1}AB$ is 1. Since $\det B^{-1}AB = \det A = ad - bc = 1$. Here, we can take $a = d = \cos\theta$ and $b = \sin\theta$, $c = -\sin\theta$. That is, $B^{-1}AB$ is a Lorentzian rotation matrix in the plane spanned by spacelike vectors $\overrightarrow{e_2}$ and $\overrightarrow{e_3}$. So, we have

$$B^{-1}AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

We see that $B^{-1}AB$ is a Lorentzian rotation matrix in the plane spanned by spacelike vectors $\overrightarrow{e_2}$ and $\overrightarrow{e_3}$, about the axis $\overrightarrow{e_1}$, by the angle θ. Moreover, we can find that eigenvalues of $B^{-1}AB$ are $\lambda_1 = 1$, $\lambda_2 = e^{i\theta} = \cos\theta + i\sin\theta$ and $\lambda_3 = e^{-i\theta} = \cos\theta - i\sin\theta$ which are also the eigenvalues of the Lorentzian rotation matrix A.

ii) Let $\overrightarrow{u_2}$ be a unit spacelike vector such that $A\overrightarrow{u_2} = \overrightarrow{u_2}$. Thus, unit spacelike vector $\overrightarrow{u_1}$ is the rotation axis of A by theorem 3. Take another unit spacelike vector $\overrightarrow{u_3}$ perpendicular to $\overrightarrow{u_2}$, then $\overrightarrow{u_1} = \overrightarrow{u_2} \wedge_L \overrightarrow{u_3}$ will be a unit timelike vector. Then $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$ is a right handed basis for $\mathbb{E}_1^3$. Let $\{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}\}$ be standard orthogonal basis of $\mathbb{E}_1^3$ and B be a matrix such that $B\overrightarrow{e_i} = \overrightarrow{u_i}$. Then, B is a pseudo orthogonal matrix because its columns are orthonormal basis for $\mathbb{E}_1^3$. Then $B^{-1}AB$ is also pseudo orthogonal. Moreover

$$B^{-1}AB\overrightarrow{e_2} = B^{-1}A\overrightarrow{u_2} = B^{-1}\overrightarrow{u_2} = \overrightarrow{e_2}.$$

Therefore, the second column of $B^{-1}AB$ is $\overrightarrow{e_2}$. The second column must be orthogonal to other two columns. So, $B^{-1}AB$ will be in the following form;

$$B^{-1}AB = \begin{bmatrix} a & 0 & c \\ 0 & 1 & 0 \\ b & 0 & d \end{bmatrix}.$$

Since $B^{-1}AB$ is also pseudo orthogonal, $-ac + bd = 0$ and $ad - bc = 1$. Here, we can take $a = d = \cosh\theta$ and $b = c = \sinh\theta$. That is, $B^{-1}AB$ is a Lorentzian rotation matrix in the plane spanned by timelike vector $\overrightarrow{e_1}$ and spacelike vector $\overrightarrow{e_3}$. So, we have

$$B^{-1}AB = \begin{bmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \end{bmatrix}.$$

We see that $B^{-1}AB$ is a Lorentzian rotation matrix in the plane spanned by timelike vector $\overrightarrow{e_1}$ and spacelike vector $\overrightarrow{e_3}$, about the axis $\overrightarrow{e_2}$, by the hyperbolic angle θ. Moreover, we can find that eigenvalues of $B^{-1}AB$ are $\lambda_1 = 1$, $\lambda_2 = e^{\theta} = \cosh\theta + \sinh\theta$ and $\lambda_3 = e^{-\theta} = \cosh\theta - \sinh\theta$ which are also the eigenvalues of the Lorentzian rotation matrix A. □

Theorem 5. *Let A be a Lorentzian rotation matrix in $\mathbb{E}_1^3$. Then, characteristic polynomial of the matrix A is $x^3 - tr\,(A)\,x^2 + tr\,(A)\,x - 1$. Further;*

i) *Eigenvalues of the A are complex and rotation axis of A is timelike vector if and only if $-1 < tr(A) < 3$,*
ii) *Eigenvalues of the A are real and rotation axis of A is spacelike vector if and only if $-1 \geq tr(A)$ or $tr(A) \geq 3$,*

where $tr(A)$ denotes the trace of the matrix A.

Proof. We know that the characteristic of a rotation matrix A in E_1^3 is

$$P(x) = \det(xI - A) = x^3 - tr(A)x^2 + Cx - 1$$

where $C \in \mathbb{R}$. Since one of the roots of characteristic polynomial is 1, $P(1) = 0$ must be satisfied. Thus we obtain $C = tr(A)$. So, the characteristic polynomial of the matrix A is $x^3 - tr\,(A)\,x^2 + tr\,(A)\,x - 1$. If we factorize this polynomial, we get

$$P(x) = (x - 1)(x^2 + (1 - tr(A)x + 1).$$

Therefore, other eigenvalues of A are the roots of the equation

$$x^2 + (1 - tr(A))x + 1 = 0.$$

The discriminant of this equation is found as

$$\Delta = (1 - tr(A))^2 - 4 = (tr(A) + 1)(tr(A) - 3).$$

$\Delta < 0$ if and only if $-1 < tr(A) < 3$. So, we find that eigenvalues of the A are complex and rotation axis of A is timelike vector if and only if $-1 < tr(A) < 3$. $\Delta \geq 0$ if and only if $-1 \geq tr(A)$ or $tr(A) \geq 3$. Thus, we obtain that eigenvalues of the A are real and rotation axis of A is spacelike vector if and only if $-1 \geq tr(A)$ or $tr(A) \geq 3$. □

Corollary 6. *Let A be a Lorentzian rotation matrix in $\mathbb{E}_1^3$. The followings are satisfied;*

i) *If rotation axis* of A is *a timelike vector, then the angle of rotation is θ* and $\cos\theta = \frac{1}{2}\left(tr\,(A) - 1\right)$,
ii) *If rotation axis* of A is *a spacelike vector, then angle of rotation is hyperbolic θ and* $\cosh\theta = \frac{1}{2}\left(tr\,(A) - 1\right)$.

Proof. If the rotation axis is a timelike vector, then the characteristic polynomial of the rotation matrix A in $\mathbb{E}_1^3$ is

$$\begin{aligned} P_A(\lambda) &= (\lambda - 1)(\lambda - e^{i\theta})(\lambda - e^{-i\theta}) \\ &= \lambda^3 - (1 + 2\cos\theta)\lambda^2 + (1 + 2\cos\theta)\lambda - 1. \end{aligned}$$

Similarly, we can find the characteristic polynomial of the rotation matrix A in $\mathbb{E}_1^3$ as

$$P_A(\lambda) = \lambda^3 - (1 + 2\cosh\theta)\lambda^2 + (1 + 2\cosh\theta)\lambda - 1$$

for the case the rotation axis is spacelike. Thus, using the previous theorem, we find $\cos\theta = \frac{1}{2}(tr(A) - 1)$ or $\cosh\theta = \frac{1}{2}(tr(A) - 1)$ according to the casual character of the rotation axis is timelike or spacelike, respectively. □

Example 7. Let's find the eigenvalues of the Lorentzian rotation matrix

$$\begin{bmatrix} 9/4 & -2 & 1/4 \\ -1 & 1 & -1 \\ -7/4 & 2 & 1/4 \end{bmatrix}$$

in $\mathbb{E}_1^3$. Trace of this matrix is $9/4+1+1/4 = 7/2 > 3$ then $2\cosh\theta+1 = 7/2$, so $\cosh\theta = 5/4$ and $\sinh\theta = 3/4$. Thus, eigenvalues of the given matrix are

$$\begin{aligned} \lambda_1 &= 1, \\ \lambda_2 &= \cosh\theta + \sinh\theta = 2, \\ \lambda_3 &= \cosh\theta - \sinh\theta = 1/2. \end{aligned}$$

Notice that these eigenvalues are real on the contrary to eigenvalues other than 1 of a rotation matrices in the Euclidean 3 space is complex.

Example 8. Let's find the eigenvalues of the rotation matrix

$$\begin{bmatrix} 15/2 & -5/2 & -7 \\ 11/2 & -5/2 & -5 \\ 5 & -1 & -5 \end{bmatrix}$$

in $\mathbb{E}_1^3$. Trace of this matrix is 0 then $2\cos\theta + 1 = 0$ and $\cos\theta = -1/2$, $\sin\theta = \sqrt{3}/2$. It means that the given represents a rotation by angle $2\pi/3$. So, the eigenvalues are

$$\begin{aligned} \lambda_1 &= 1, \\ \lambda_2 &= \cos\theta + i\sin\theta = -1/2 + i\sqrt{3}/2 \\ \lambda_3 &= \cos\theta - i\sin\theta = -1/2 - i\sqrt{3}/2. \end{aligned}$$

Theorem 9. *Let $q = (q_1, q_2, q_3, q_4)$ be a unit timelike split quaternion, then eigenvalues of the rotation matrix which is generated by q are*

$$1, \quad \left(|q_1| - \sqrt{q_1^2 - 1}\right)^2 \quad \textit{and} \quad \left(|q_1| + \sqrt{q_1^2 - 1}\right)^2.$$

That is, eigenvalues depends only the first component of the q.

Proof. Let R be the rotation matrix generated by unit timelike quaternion $q = (q_1, q_2, q_3, q_4)$. By considering equation 1, we find the eigenvalues of R as follows;

$$\begin{aligned} \lambda_1 &= q_1^2 + q_2^2 - q_3^2 - q_4^2, \\ \lambda_2 &= q_1^2 - q_2^2 + q_3^2 + q_4^2 - 2\sqrt{q_1^2 q_3^2 - q_1^2 q_2^2 + q_1^2 q_4^2}, \\ \lambda_3 &= q_1^2 - q_2^2 + q_3^2 + q_4^2 + 2\sqrt{q_1^2 q_3^2 - q_1^2 q_2^2 + q_1^2 q_4^2}. \end{aligned}$$

Since, q is unit timelike quaternion, $q_1^2 + q_2^2 - q_3^2 - q_4^2 = 1$. Using this equality, we obtain the eigenvalues as;

$$\lambda_1 = 1, \ \lambda_2 = \left(|q_1| - \sqrt{q_1^2 - 1}\right)^2, \ \lambda_3 = \left(|q_1| + \sqrt{q_1^2 - 1}\right)^2. \qquad \square$$

Remark 10. Trace of the R given in equation 1 is $4q_1^2 - 1$. On the other hand, characteristic polynomial of the R can be find as $P(\lambda) = \lambda^3 + \lambda^2 \left(1 - 4q_1^2\right) + \lambda \left(4q_1^2 - 1\right) - 1$ by using theorem 5. The rotation axis is timelike if and only if

$$\Delta = \left(4q_1^2 - 2\right)^2 - 4 = 16q_1^2 \left(q_1^2 - 1\right) < 0.$$

The rotation axis is spacelike if and only if

$$\Delta = \left(4q_1^2 - 2\right)^2 - 4 = 16q_1^2 \left(q_1^2 - 1\right) \geq 0.$$

That is, the rotation axis is timelike if and only if $q_1 \in (-1, 0) \cup (0, 1)$ and otherwise, the rotation axis is spacelike. So, casual character of the rotation axis of the rotation matrix R depends only first component of the unit timelike split quaternion corresponding to R. In the case $q_1 = 0$, the eigenvalues of R are $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -1$, and the eigenvectors corresponding to these eigenvalues are;

$$\overrightarrow{u_1}(0, q_2, q_3, q_4) = \begin{bmatrix} q_2 \\ q_3 \\ q_4 \end{bmatrix}, \ \overrightarrow{c_2}(0, q_2, q_3, q_4) = \begin{bmatrix} q_3 \\ q_2 \\ 0 \end{bmatrix}, \ \overrightarrow{c_3}(0, q_2, q_3, q_4) = \begin{bmatrix} q_4 \\ 0 \\ q_2 \end{bmatrix},$$

respectively. Here, the rotation axis is a timelike vector, since $\langle \overrightarrow{u_1}, \overrightarrow{u_1} \rangle_L = -q_2^2 + q_3^2 + q_4^2 = -1 < 0$. Thus, the rotation axis is timelike if and only if $q_1 \in (-1, 1)$, the rotation axis is spacelike if and only if $q_1 \in \mathbb{R} - \{[-1, 1]\}$.

Theorem 11. *For a rotation matrix R given in equation 1,*

i) *If $q_1 \neq 0$, then the eigenvectors of R other than rotation axis are null vectors,*
ii) *If $q_1 = 0$, then the eigenvectors of R other than rotation axis are spacelike vectors.*

Proof. Let R be a rotation matrix given in equation 1.

i) Suppose $q_1 \neq 0$. We can find the eigenvectors of R with long and tedious computations as follows:

$$\overrightarrow{c_2}(q_1, q_2, q_3, q_4) = \begin{bmatrix} q_1^2 q_2 q_4 + q_3 q_1 \left(q_1^2 - 1\right) - \left(q_1 q_3 + q_2 q_4\right) \sqrt{q_1^4 - q_1^2} \\ q_1^2 q_3 q_4 + q_2 q_1 \left(q_1^2 - 1\right) - \left(q_1 q_2 + q_3 q_4\right) \sqrt{q_1^4 - q_1^2} \\ \left(q_1^2 - \sqrt{q_1^4 - q_1^2}\right)\left(q_2^2 - q_3^2\right) \end{bmatrix} \tag{2}$$

for the eigenvalue $e^{-i\theta}$ (or $e^{-\theta}$) and

$$\overrightarrow{c_3}(q_1, q_2, q_3, q_4) = \begin{bmatrix} -q_1^2 q_2 q_4 - q_3 q_1 \left(q_1^2 - 1\right) - \left(q_1 q_3 + q_2 q_4\right) \sqrt{q_1^4 - q_1^2} \\ -q_1^2 q_3 q_4 - q_2 q_1 \left(q_1^2 - 1\right) - \left(q_1 q_2 + q_3 q_4\right) \sqrt{q_1^4 - q_1^2} \\ \left(-q_1^2 - \sqrt{q_1^4 - q_1^2}\right)\left(q_2^2 - q_3^2\right) \end{bmatrix} \tag{3}$$

for the eigenvalue $e^{i\theta}$ (or e^{θ}). Therefore, one can prove that c_2 and c_3 are null vectors using the equality $q_1^2 + q_2^2 - q_3^2 - q_4^2 = 1$.

ii) Suppose $q_1 = 0$. By remark 10, the eigenvectors of R other than rotation axis are

$$\overrightarrow{c_2}(0, q_2, q_3, q_4) = \begin{bmatrix} q_3 \\ q_2 \\ 0 \end{bmatrix}, \ \overrightarrow{c_3}(0, q_2, q_3, q_4) = \begin{bmatrix} q_4 \\ 0 \\ q_2 \end{bmatrix}.$$

Since $\langle \overrightarrow{c_2}, \overrightarrow{c_2} \rangle_L = -q_3^2 + q_2^2 = 1 + q_4^2 > 0$ and $\langle \overrightarrow{c_3}, \overrightarrow{c_3} \rangle_L = -q_4^2 + q_2^2 = 1 + q_3^2 > 0$, the eigenvectors of R other than rotation axis are spacelike vectors. □

Example 12. Let's find eigenvectors of the matrix

$$R(3, 0, 2, 2) = \begin{bmatrix} 17 & 12 & -12 \\ 12 & 9 & -8 \\ -12 & -8 & 9 \end{bmatrix}.$$

Using the above theorem, we find the eigenvectors

$$\overrightarrow{c_2}(3, 0, 2, 2) = \begin{bmatrix} -36\sqrt{2} + 48 \\ -24\sqrt{2} + 36 \\ 24\sqrt{2} - 36 \end{bmatrix}, \ \overrightarrow{c_3}(3, 0, 2, 2) = \begin{bmatrix} -36\sqrt{2} - 48 \\ -24\sqrt{2} - 36 \\ 24\sqrt{2} + 36 \end{bmatrix}$$

corresponding to the eigenvalues $17 - 12\sqrt{2}$ and $12\sqrt{2} + 17$, respectively. If we divide the vectors $\overrightarrow{c_2}$ and $\overrightarrow{c_3}$ by their third components, we find $\overrightarrow{v_2} = \left(\sqrt{2}, -1, 1\right)$ and $\overrightarrow{v_3} = \left(-\sqrt{2}, -1, 1\right)$. Notice that the vectors $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$ are null vectors.

Remark 13. Since the eigenvectors other than rotation axis are null vectors for the case $q_1 \neq 0$, eigenvectors of a rotation matrix does not form a basis for three dimensional Minkowski 3 space. But using these eigenvectors we can find an orthogonal basis for $\mathbb{E}_1^3$. Moreover, the eigenvectors other than rotation axis are spacelike vectors in case $q_1 = 0$. So, we may also find an orthogonal basis for $\mathbb{E}_1^3$ by using the eigenvectors for the case $q_1 = 0$.

Theorem 14. *Let $\overrightarrow{u_1}$, $\overrightarrow{c_2}$ and $\overrightarrow{c_3}$ be the eigenvectors of the a Lorentzian rotation matrix R and the rotation axis $\overrightarrow{u_1}$ be a timelike vector. Then the following are satisfied;*

i) *If $q_1 \neq 0$, then the real vectors,*

$$\overrightarrow{u_2} = \frac{i}{2}\left(\overrightarrow{c_2} + \overrightarrow{c_3}\right) \quad \text{and} \quad \overrightarrow{u_3} = \frac{1}{2}\left(\overrightarrow{c_2} - \overrightarrow{c_3}\right)$$

are spacelike. Also, the vectors $\overrightarrow{u_1}$, $\overrightarrow{u_2}$, $\overrightarrow{u_3}$ are orthogonal to each other. So, they form an orthogonal basis for E_1^3.

ii) *If $q_1 = 0$, then the real vectors,*

$$\overrightarrow{u_2} = \overrightarrow{c_2}, \ \overrightarrow{u_3} = \overrightarrow{c_1} \wedge_L \overrightarrow{c_2} \text{ and } \overrightarrow{v_2} = \overrightarrow{c_3}, \ \overrightarrow{v_3} = \overrightarrow{c_1} \wedge_L \overrightarrow{c_3}$$

are spacelike. $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$ and $\{\overrightarrow{u_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ are two different orthogonal basis for E_1^3.

Proof. Suppose that $\overrightarrow{u_1}$, $\overrightarrow{c_2}$ and $\overrightarrow{c_3}$ are the eigenvectors of the a Lorentzian rotation matrix R and the rotation axis $\overrightarrow{u_1}$ is a timelike vector.

i) Assume that $q_1 \neq 0$. According to the equalities 2 and 3, we get

$$\overrightarrow{u_2} = \begin{bmatrix} -i\left(q_1q_3+q_2q_4\right)\sqrt{-q_1^2+q_1^4} \\ -i\left(q_1q_2+q_3q_4\right)\sqrt{-q_1^2+q_1^4} \\ i\left(q_3^2-q_2^2\right)\sqrt{q_1^4-q_1^2} \end{bmatrix} \text{ and } \overrightarrow{u_3} = \begin{bmatrix} q_1^2q_2q_4+q_1q_3\left(q_1^2-1\right) \\ q_1^2q_3q_4+q_1q_2\left(q_1^2-1\right) \\ q_1^2\left(q_2^2-q_3^2\right) \end{bmatrix}$$

where $< \overrightarrow{u_2}, \overrightarrow{u_3} >_L = 0$. Moreover, if the rotation axis is timelike then $q_1 \in (-1,0) \cup (0,1)$. Therefore, from the equality

$$< \overrightarrow{u_2}, \overrightarrow{u_2} >_L = q_1^2\left(q_1^2-1-q_4^2\right)\left(q_1^2-1\right),$$

we get $< \overrightarrow{u_2}, \overrightarrow{u_2} >_L > 0$. It means that $\overrightarrow{u_2}$ is a spacelike vector. Similarly, one can find

$$< \overrightarrow{u_3}, \overrightarrow{u_3} >_L = q_1^2\left(q_1^2-1-q_4^2\right)\left(q_1^2-1\right).$$

That is, $\overrightarrow{u_3}$ is also a spacelike vector.

ii) This is the direct result of remark 10 and theorem 11. □

Example 15. Let's take the rotation matrix generated by $(1/2, 1, 1/2, 0)$. So,

$$R(1/2,1,1/2,0) = \begin{bmatrix} 3/2 & -1 & -1/2 \\ 1 & -1 & -1 \\ -1/2 & 1 & -1/2 \end{bmatrix}.$$

As $q_1 = 1/2$, rotation axis is timelike vector $\overrightarrow{u_1} = (1, 1/2, 0)$. Other eigenvectors of this matrix are

$$\overrightarrow{c_2}(1/2,1,1/2,0) = \begin{bmatrix} -3/16 - i\sqrt{3}/16 \\ -3/8 - i\sqrt{3}/8 \\ 3/16 - i3\sqrt{3}/16 \end{bmatrix},$$

$$\overrightarrow{c_3}(1/2,1,1/2,0) = \begin{bmatrix} 3/16 - i\sqrt{3}/16 \\ 3/8 - i\sqrt{3}/8 \\ -3/16 - i3\sqrt{3}/16 \end{bmatrix}.$$

These vectors are null vectors. Therefore, we get the spacelike vectors

$$\overrightarrow{u_2} = \frac{i}{2}\left(\overrightarrow{c_2}+\overrightarrow{c_3}\right) = \begin{bmatrix} \sqrt{3}/16 \\ \sqrt{3}/8 \\ 3\sqrt{3}/16 \end{bmatrix} \text{ and } \overrightarrow{u_3} = \frac{1}{2}\left(\overrightarrow{c_2}-\overrightarrow{c_3}\right) = \begin{bmatrix} -3/16 \\ -3/8 \\ 3/16 \end{bmatrix}.$$

Thus, we obtain an orthogonal basis

$$\left\{(1,1/2,0), \left(\sqrt{3}/16, \sqrt{3}/8, 3\sqrt{3}/16\right), (-3/16,-3/8,3/16)\right\}$$

for the Minkowski 3 space.

Theorem 16. *Let $\overrightarrow{u_2}$, $\overrightarrow{c_2}$ and $\overrightarrow{c_3}$ be the eigenvectors of the a Lorentzian rotation matrix A and the rotation axis $\overrightarrow{u_2}$ be a spacelike vector.*

i) *If $q_1^2 - q_4^2 > 1$, then $\overrightarrow{u_1} = \frac{1}{2}\left(\overrightarrow{c_2}+\overrightarrow{c_3}\right)$ is a timelike vector and $\overrightarrow{u_3} = \frac{1}{2}\left(\overrightarrow{c_2}-\overrightarrow{c_3}\right)$ is a spacelike vector,*

ii) *If* $q_1^2 - q_4^2 < 1$, *then* $\overrightarrow{u_1} = \frac{1}{2}(\overrightarrow{c_2} + \overrightarrow{c_3})$ *is a spacelike vector and* $\overrightarrow{u_3} = \frac{1}{2}(\overrightarrow{c_2} - \overrightarrow{c_3})$ *is a timelike vector.*

For each case, the vectors $\overrightarrow{u_1}$, $\overrightarrow{u_2}$ *and* $\overrightarrow{u_3}$ *are orthogonal to each other. So, they form an orthogonal basis for* E_1^3.

Proof. According to the equalities 2 and 3, we get

$$\overrightarrow{u_1} = \begin{bmatrix} -(q_1q_3 + q_2q_4)\sqrt{q_1^2(q_1^2-1)} \\ -(q_1q_2 + q_3q_4)\sqrt{q_1^2(q_1^2-1)} \\ -(q_2^2 - q_3^2)\sqrt{q_1^2(q_1^2-1)} \end{bmatrix} \text{ and } \overrightarrow{u_3} = \begin{bmatrix} q_1^2q_2q_4 + q_1q_3(q_1^2-1) \\ q_1^2q_3q_4 + q_1q_2(q_1^2-1) \\ q_1^2(q_2^2 - q_3^2) \end{bmatrix},$$

where $< \overrightarrow{u_1}, \overrightarrow{u_3} >_L = 0$. And we have the following equalities;

$$< \overrightarrow{u_1}, \overrightarrow{u_1} >_L = -q_1^2(q_1^2 - 1 - q_4^2)(q_1^2 - 1),$$
$$< \overrightarrow{u_3}, \overrightarrow{u_3} >_L = q_1^2(q_1^2 - 1 - q_4^2)(q_1^2 - 1).$$

Using $q_1 \in \mathbb{R} - \{[-1,1]\}$, we have the following cases;

i) If $q_1^2 - q_4^2 > 1$, then $< \overrightarrow{u_1}, \overrightarrow{u_1} >_L < 0$ and $< \overrightarrow{u_3}, \overrightarrow{u_3} >_L > 0$. Thus $\overrightarrow{u_1}$ is a timelike vector and $\overrightarrow{u_3}$ is a spacelike vector,

ii) If $q_1^2 - q_4^2 < 1$, then $< \overrightarrow{u_1}, \overrightarrow{u_1} >_L > 0$ and $< \overrightarrow{u_3}, \overrightarrow{u_3} >_L < 0$. Thus $\overrightarrow{u_1}$ is a spacelike vector and $\overrightarrow{u_3}$ is a timelike vector.

Consequently, $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$ is an orthogonal basis for $\mathbb{E}_1^3$. □

Example 17. Let us find an orthogonal basis for $\mathbb{E}_1^3$ by using eigenvectors of Lorentzian rotation matrix

$$R(5,1,4,3) = \begin{bmatrix} 51 & 22 & -46 \\ 38 & 17 & -34 \\ -34 & -14 & 31 \end{bmatrix}.$$

Since $q_1 = 5 \in \mathbb{R} - \{[-1,1]\}$, the rotation axis

$$\overrightarrow{u_2}(5,1,3,4) = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

is spacelike. By using the proof of theorem 16, the other vectors are found as follows;

$$\overrightarrow{u_1}(5,1,3,4) = \begin{bmatrix} -190\sqrt{6} \\ -170\sqrt{6} \\ 80\sqrt{6} \end{bmatrix}, \ \overrightarrow{u_3} = \begin{bmatrix} 460 \\ 420 \\ -200 \end{bmatrix}.$$

It can be seen that these three vectors are orthogonal to each other. Since

$$\langle \overrightarrow{u_1}, \overrightarrow{u_1} \rangle_L = -4800 < 0 \text{ and } \langle \overrightarrow{u_3}, \overrightarrow{u_3} \rangle_L = 4800 > 0,$$

$\overrightarrow{u_1}$ is a timelike vector and $\overrightarrow{u_2}$ is a spacelike vector. This is a consequence of theorem 16, because $q_1^2 - q_4^2 = 25 - 9 > 1$. The set $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$ is an orthogonal basis for $\mathbb{E}_1^3$.

Example 18. Let us find an orthogonal basis for $\mathbb{E}_1^3$ by using eigenvectors of Lorentzian rotation matrix obtained by unit timelike split quaternion $q = (5, 5, 0, 7)$.Since $q_1 = 5 \in \mathbb{R} - \{[-1, 1]\}$, the rotation axis

$$\overrightarrow{u_2}(5,5,0,7) = \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}$$

is spacelike. We can find

$$\overrightarrow{u_1}(5,5,0,7) = \begin{bmatrix} -350\sqrt{6} \\ -250\sqrt{6} \\ -250\sqrt{6} \end{bmatrix}, \ \overrightarrow{u_3}(5,5,0,7) = \begin{bmatrix} 875 \\ 600 \\ 625 \end{bmatrix}$$

by proof of theorem 16. It can be seen that these three vectors are orthogonal to each other. We have

$$\langle \overrightarrow{u_1}, \overrightarrow{u_1} \rangle_L = 15000 > 0 \text{ and } \langle \overrightarrow{u_3}, \overrightarrow{u_3} \rangle_L = -15000 > 0.$$

Thus, $\overrightarrow{u_1}$ is a spacelike vector and $\overrightarrow{u_2}$ is a timelike vector. This is also a result of theorem 16, because $q_1^2 - q_4^2 = 25 - 49 < 1$. The set $\{\overrightarrow{u_1}, \overrightarrow{u_2}, \overrightarrow{u_3}\}$ is an orthogonal basis for $\mathbb{E}_1^3$.

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