

## On focal curves of null Cartan curves

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**Abstract:** The focal curve, which is determined as the locus of centers of osculating pseudo-spheres of a null Cartan curve, is investigated in Minkowski  $(n+2)$ -space  $\mathcal{M}^{n+2}$ . Moreover, a curve called *acceleration focal curve* of a null Cartan curve is introduced by using a new family of functions.

**Key words:** Focal curve, null Cartan curve, vertex point, A-focal curve

### 1. Introduction

The *focal set* (or *caustic*) of a curve or a surface in Euclidean 3-space  $\mathbb{R}^3$  is the locus of its centers of curvature. The focal sets are of great importance in singularity theory. They can be used as an interrogation tool in order to study the singular points of a curve or surface. The points of focal set for a curve correspond to the centers of its osculating spheres. Thus, the *focal curve* of a smooth curve in  $\mathbb{R}^n$  is defined as the locus of points corresponding to the centers of its osculating hyperspheres. Vargas [18] studied the geometry of focal sets, focusing on the properties of the focal curves. Using these properties, he formulated and proved new results for curves in Euclidean  $n$ -space for arbitrary  $n \geq 2$ .

In Lorentzian geometry, there are various studies related to focal sets. The geometry of the focal set of a smooth surface in  $\mathcal{M}^3$  was studied by Tari [17] using the family of distance squared functions and by Şimşek and Özdemir [16] in terms of line congruences. On the other hand, by means of the *volumelike distance function* given by  $D : I \times \mathcal{M}^3 \rightarrow \mathbb{R}$ ,  $D(s, v) = (\gamma(s) - \mathbf{v}) \cdot \mathbf{N}$ , where  $\{\mathbf{L}, \mathbf{N}, \mathbf{W}\}$  is a null Cartan frame of  $\gamma$ , Wang et al. [19] defined a surface  $FS : I \times \mathbb{R} \rightarrow \mathcal{M}^3$  and a curve  $F_\gamma : I \rightarrow \mathcal{M}^3$  for a null Cartan curve  $\gamma : I \rightarrow \mathcal{M}^3$  as the following

$$FS(s, \mu) = \gamma(s) + \frac{1}{k(s)} \mathbf{W}(s) + \mu \mathbf{N}(s), \quad F_\gamma(s) = \gamma(s) + \frac{1}{k(s)} \mathbf{W}(s), \quad (1)$$

respectively, where  $k(s)$  is a curvature function of  $\gamma$ . They called  $FS$  and  $F_\gamma$  the focal surface and focal curve of null Cartan curve  $\gamma$ ; however,  $F_\gamma$  is not the locus of the centers of osculating spheres of  $\gamma$ . Actually, Izumiya [10] introduced the volumelike distance function (or binormal directed distance function) in order to study singularities of certain surfaces and curves associated with the family of rectifying planes along a space curve in  $\mathbb{R}^3$ . Liu and Wang [11] classified the singularities of lightlike hypersurfaces and lightlike focal sets, which are generated by null Cartan curves, by using the lightlike distance squared function in Minkowski space-time.

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They stated that the lightlike focal set of a null Cartan curve corresponds to the locus of centers of its osculating pseudo-sphere having five-point contact with the null Cartan curve. For a spacelike and timelike curve, the properties of focal curves were studied by Özdemir [14].

Bonnor [2] introduced the Cartan frame to study the behaviors of a null curve and proved the fundamental existence and congruence theorems in Minkowski space-time. Bejancu [1] presented a method for the general study of the geometry of null curves in Lorentz manifolds and, more generally, in semi-Riemannian manifolds (see also the book [5]). Ferrandez et al. [6] gave a reference along a null curve in an n-dimensional Lorentzian space. They showed the fundamental existence and uniqueness theorems and described the null helices in higher dimensions. Cöken and Çiftçi [4] characterized the pseudo-spherical null curves and Bertrand null curves in the Minkowski space-time.

The study of the geometry of null curves has become of growing importance in mathematical physics. The null curves are useful to find the solution of some equations in classical relativistic string theory (see [3, 8, 9]). Moreover, there exists a geometric particle model associated with the geometry of null curves in the Minkowski space-time (see [7, 12]).

The paper is organized as follows. First, we give basic information about null Cartan curves. Then we investigate the focal curve and focal curvatures of a null Cartan curve. We also give necessary and sufficient conditions in order that a point of a null Cartan curve is a vertex and the null Cartan curve is pseudo-spherical in  $\mathcal{M}^{n+2}$ . In the next section, we define the acceleration focal curve of a null Cartan curve by using a new family of functions called null acceleration directed distance function. Moreover, we examine some geometric properties of the acceleration focal curve of a null Cartan curve.

## 2. Preliminaries

Let  $\mathbf{u} = (u_1, u_2, \dots, u_{n+2})$ ,  $\mathbf{v} = (v_1, v_2, \dots, v_{n+2})$  be two arbitrary vectors in Minkowski space  $\mathcal{M}^{n+2}$ . The Lorentzian inner product of  $\mathbf{u}$  and  $\mathbf{v}$  can be stated as  $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} I^* \mathbf{v}^T$ , where  $I^* = \text{diag}(-1, 1, \dots, 1)$ . We say that a vector  $\mathbf{u}$  in  $\mathcal{M}^{n+2}$  is called spacelike, null (lightlike), or timelike if  $\mathbf{u} \cdot \mathbf{u} > 0$ ,  $\mathbf{u} \cdot \mathbf{u} = 0$ , or  $\mathbf{u} \cdot \mathbf{u} < 0$ , respectively. The norm of the vector  $\mathbf{u}$  is represented by  $\|\mathbf{u}\| = \sqrt{|\mathbf{u} \cdot \mathbf{u}|}$ .

We can describe the pseudo-spheres in  $\mathcal{M}^{n+2}$  as follows: The *hyperbolic (n+1)-space* is defined by

$$H^{n+1}(-1) = \{\mathbf{u} \in \mathcal{M}^{n+2} : \mathbf{u} \cdot \mathbf{u} = -1\}$$

and *de Sitter (n+1)-space* is defined by

$$S_1^{n+1} = \{\mathbf{u} \in \mathcal{M}^{n+2} : \mathbf{u} \cdot \mathbf{u} = 1\} \text{ (see [13]).}$$

A basis  $\mathbf{B} = \{\mathbf{L}, \mathbf{N}, \mathbf{W}_1, \dots, \mathbf{W}_n\}$  is said to pseudo-orthonormal if it satisfies the following conditions:

$$\begin{aligned} \mathbf{L} \cdot \mathbf{L} &= \mathbf{N} \cdot \mathbf{N} = 0, & \mathbf{L} \cdot \mathbf{N} &= 1 \\ \mathbf{L} \cdot \mathbf{W}_i &= \mathbf{N} \cdot \mathbf{W}_i = \mathbf{W}_i \cdot \mathbf{W}_j = 0 \end{aligned}$$

$$\mathbf{W}_i \cdot \mathbf{W}_i = 1$$

for  $i, j = 1, \dots, n$  and  $i \neq j$  ([5]).

A curve locally parameterized by  $\gamma : J \subset \mathbb{R} \rightarrow \mathcal{M}^{n+2}$  is called a null curve if  $\gamma'(t) \neq 0$  is a null vector for all  $t$ . We know that a null curve  $\gamma(t)$  satisfies  $\gamma''(t) \cdot \gamma''(t) \geq 0$  (see [5]). If  $\gamma''(t) \cdot \gamma''(t) = 1$ , then it is said that a null curve  $\gamma(t)$  in  $\mathcal{M}^{n+2}$  is parameterized by *pseudo-arc*. If we assume that the acceleration vector of the null curve is not null, the pseudo-arc parametrization becomes as follows:

$$s = \int_{t_0}^t (\gamma''(u) \cdot \gamma''(u))^{1/4} du \quad ([2, 4]). \tag{2}$$

A null curve  $\gamma(t)$  in  $\mathcal{M}^{n+2}$  with  $\gamma''(t) \cdot \gamma''(t) \neq 0$  is a *Cartan curve* if

$$S_\gamma := \{\gamma'(t), \gamma''(t), \dots, \gamma^{(n+2)}(t)\}$$

is linearly independent for any  $t$ . There exists a unique *Cartan frame*  $C_\gamma := \{\mathbf{L}, \mathbf{N}, \mathbf{W}_1, \dots, \mathbf{W}_n\}$  of the Cartan curve that has the same orientation with  $S_\gamma$  according to pseudo arc-parameter  $t$ , such that the following equations are satisfied:

$$\begin{aligned} \gamma' &= \mathbf{L}, \\ \mathbf{L}' &= \mathbf{W}_1, \\ \mathbf{N}' &= k_1 \mathbf{W}_1 + k_2 \mathbf{W}_2, \\ \mathbf{W}'_1 &= -k_1 \mathbf{L} - \mathbf{N}, \\ \mathbf{W}'_2 &= -k_2 \mathbf{L} + k_3 \mathbf{W}_3, \\ \mathbf{W}'_i &= -k_i \mathbf{W}_{i-1} + k_{i+1} \mathbf{W}_{i+1}, \quad i \in \{3, \dots, n-1\} \\ \mathbf{W}'_n &= -k_n \mathbf{W}_{n-1}, \end{aligned} \tag{3}$$

where  $\mathbf{N} = -\gamma^{(3)} - \frac{1}{2}(\gamma^{(3)} \cdot \gamma^{(3)})\gamma'$  is a null vector, which is called a null transversal vector field, and  $C_\gamma$  is pseudo-orthonormal and positively oriented. The functions  $k_i, i \in \{1, \dots, n\}$ , are called *Cartan curvatures* of  $\gamma$ . The papers [1, 2, 5, 6] can be seen for more information about the geometry of null curves.

Let  $f : \mathcal{M}^{n+2} \rightarrow \mathcal{M}^{n+2}$  be a differentiable function and let  $\gamma : I \rightarrow \mathcal{M}^{n+2}$  be a null Cartan curve. We state that  $\gamma$  and  $f^{-1}(0)$  have *k-point contact* for  $t = t_0$  if the function  $g(t) = f \circ \gamma(t)$  satisfies  $g(t_0) = g'(t_0) = \dots = g^{(k-1)}(t_0) = 0, g^{(k)}(t_0) \neq 0$ . If we just have the condition  $g(t_0) = g'(t_0) = \dots = g^{(k-1)}(t_0) = 0$ , then it is said that  $\gamma$  and  $f^{-1}(0)$  have *at least k-point contact* for  $t = t_0$ .

### 3. The focal curves of null Cartan curves

Let  $\gamma : I \rightarrow \mathcal{M}^{n+2}$  be a null Cartan curve. For  $k = 3, \dots, n + 1$ , a *k-osculating pseudo-sphere* at a point of  $\gamma$  in  $\mathcal{M}^{n+2}$  is a k-dimensional Lorentzian sphere having at least (k + 2)-point contact with the curve at that point. For  $k = n + 1$ , it is called the *osculating pseudo-sphere*. In this section, we consider the *lightlike distance squared function*  $f : I \times \mathcal{M}^{n+2} \rightarrow \mathbb{R}$ , which is useful for studying the focal curve of null Cartan curve, defined by

$$f(t, p) = (p - \gamma(t)) \cdot (p - \gamma(t)) - r^2, \tag{4}$$

where  $r$  is the radius of the osculating pseudo-sphere. Throughout this section, we assume  $n \geq 2$ .

**Definition 1** A vertex of null Cartan curve is a point at which the curve has at least  $(n+3)$ -point contact with its osculating pseudo-sphere (see [18] for Euclidean space).

**Definition 2** The focal curve  $F_\gamma : t \in I \rightarrow F_\gamma(t) \in \mathcal{M}^{n+2}$  of  $\gamma$  is the locus of the centers of its osculating pseudo-spheres.

**Remark 3** For any  $t_0 \in I$ , the position vector  $F_\gamma(t) - \gamma(t)$  can be written in the form

$$F_\gamma(t_0) - \gamma(t_0) = a(t_0)\mathbf{L}(t_0) + b(t_0)\mathbf{N}(t_0) + c_1(t_0)\mathbf{W}_1(t_0) + \dots + c_n(t_0)\mathbf{W}_n(t_0),$$

where  $a, b$ , and  $c_i, 1 \leq i \leq n$  are differentiable functions on  $\mathbb{R}$ . If we denote  $f_{p_0}(t) = f(t, p_0)$ , where  $p_0 = F_\gamma(t_0)$ , then the equations

$$f_{p_0}(t_0) = f'_{p_0}(t_0) = f''_{p_0}(t_0) = \dots = f^{(n+2)}_{p_0}(t_0) = 0$$

are satisfied from the definition. We can get  $a = b = c_1 = 0$  by using the equations  $f'_{p_0}(t_0) = f''_{p_0}(t_0) = f'''_{p_0}(t_0) = 0$ . Then we can state the focal curve  $F_\gamma$  of the null Cartan curve  $\gamma$  as the following:

$$F_\gamma = \gamma + c_2\mathbf{W}_2 + c_3\mathbf{W}_3 + \dots + c_n\mathbf{W}_n. \tag{5}$$

Thus, the focal curve of  $\gamma$  is determined in  $\mathcal{M}^{n+2}$  for  $n \geq 2$  with respect to definition 2.

**Definition 4** The coefficients  $c_i, i = 2, \dots, n$ , are called the  $i^{th}$  focal curvatures of null Cartan curve  $\gamma$ .

**Lemma 5** Let  $\gamma : I \rightarrow \mathcal{M}^{n+2}$  be a null Cartan curve. Then the velocity vector of the focal curve of  $\gamma$  is proportional to the spacelike Frenet vector  $\mathbf{W}_n$  of  $\gamma$ .

**Proof** Equation (4) can be stated as follows:

$$-f_p = -F_\gamma \cdot F_\gamma + 2F_\gamma \cdot \gamma - 2g - r^2,$$

where  $g = (\gamma \cdot \gamma) / 2$ . Definition 2 implies that

$$\begin{aligned} \gamma' \cdot F_\gamma(t) - g' &= 0, \\ \gamma'' \cdot F_\gamma(t) - g'' &= 0, \\ &\vdots \\ \gamma^{(n+2)} \cdot F_\gamma(t) - g^{(n+2)} &= 0. \end{aligned} \tag{6}$$

Taking the derivative of these equations, we obtain

$$\begin{aligned} \gamma' \cdot F'_\gamma(t) + \gamma'' \cdot F_\gamma(t) - g'' &= 0, \\ \gamma'' \cdot F'_\gamma(t) + \gamma''' \cdot F_\gamma(t) - g''' &= 0, \\ &\vdots \\ \gamma^{(n+2)} \cdot F'_\gamma(t) + \gamma^{(n+3)} \cdot F_\gamma(t) - g^{(n+3)} &= 0, \end{aligned} \tag{7}$$

respectively. Combining the  $i^{th}$  equation of system (6) with the  $(i+1)^{th}$  equation of system (7), we arrive at

$$\begin{aligned} \gamma' \cdot F'_\gamma(t) &= 0, \\ \gamma'' \cdot F'_\gamma(t) &= 0, \\ &\vdots \\ \gamma^{(n+1)} \cdot F'_\gamma(t) &= 0, \end{aligned} \tag{8}$$

which completes the proof. □

**Corollary 6** *The focal curve of a null Cartan curve in  $\mathcal{M}^{n+2}$  is always a spacelike curve.*

**Lemma 7** *There is a vertex point on a null Cartan curve in  $\mathcal{M}^{n+2}$  if and only if the velocity vector of its focal curve is zero at this point.*

**Proof** First, we consider a point  $\gamma(t_0)$  is a vertex of null Cartan curve  $\gamma$ . Then it satisfies the equation

$$\gamma^{(n+3)} \cdot F_\gamma(t_0) - g^{(n+3)} = 0$$

such that the last equation of (7) gives the equation  $\gamma^{(n+2)} \cdot F'_\gamma(t_0) = 0$ . This equation together with the system (8) implies that  $F'_\gamma(t_0)$  is zero.

Conversely, let us assume that  $F_\gamma(t_0) = 0$  and  $\gamma(t_0)$  is not a vertex. The corresponding point of focal curve satisfies the relation

$$\gamma^{(n+3)}(t_0) \cdot F_\gamma(t_0) - g^{(n+3)}(t_0) \neq 0.$$

From the last equation of (7), we obtain  $F'_\gamma(t_0) \neq 0$ , which implies the contradiction. □

The next theorem shows the focal curvatures of a null Cartan curve satisfy a system of *scalar Frenet equations*, which is obtained from the usual Euclidean Frenet equations in  $\mathbb{R}^n$  by changing the Frenet vectors with the focal curvatures.

**Theorem 8** *The focal curvatures of a null Cartan curve  $\gamma$  satisfy the following “scalar Frenet equations” for  $c_n \neq 0$ :*

$$\begin{pmatrix} 1 \\ c'_2 \\ c'_3 \\ c'_4 \\ \vdots \\ c'_{n-2} \\ c'_{n-1} \\ c'_n - \frac{(r_n^2)'}{2c_n} \end{pmatrix} = \begin{pmatrix} 0 & k_2 & 0 & \cdots & 0 & 0 & 0 \\ -k_2 & 0 & k_3 & \cdots & 0 & 0 & 0 \\ 0 & -k_3 & 0 & & & & \vdots \\ 0 & 0 & -k_4 & & & & \vdots \\ \vdots & 0 & & & & & \vdots \\ 0 & & & & 0 & k_{n-1} & 0 \\ 0 & 0 & & \cdots & -k_{n-1} & 0 & k_n \\ 0 & 0 & & & 0 & -k_n & 0 \end{pmatrix} \begin{pmatrix} 0 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \\ c_{n-2} \\ c_{n-1} \\ c_n \end{pmatrix}. \tag{9}$$

**Proof** If we derive the focal curve  $F_\gamma$  defined by (5) with respect to pseudo-arc length parameter and use the Frenet equations of  $\gamma$ , then we get

$$\begin{aligned} F'_\gamma &= (1 - c_2k_2) \mathbf{L} + (c'_2 - k_3c_3) \mathbf{W}_2 + (c'_3 + c_2k_3 - c_4k_4) \mathbf{W}_3 + \cdots \\ &\quad + (c'_{n-1} + c_{n-2}k_{n-1} - c_nk_n) \mathbf{W}_{n-1} + (c'_n + k_n c_{n-1}) \mathbf{W}_n. \end{aligned}$$

Lemma 5 gives rise to  $F'_\gamma = (c'_n + k_n c_{n-1}) \mathbf{W}_n$  and the following equalities are valid:

$$\begin{aligned} 1 &= c_2 k_2 \\ c'_2 &= c_3 k_3 \\ c'_3 &= -c_2 k_3 + c_4 k_4 \\ &\vdots \\ c'_{n-1} &= -c_{n-2} k_{n-1} + c_n k_n. \end{aligned}$$

Moreover, derivating the equation  $r_n^2 = (F_\gamma - \gamma) \cdot (F_\gamma - \gamma)$ , we get

$$\begin{aligned} (r_n^2)' &= 2(F'_\gamma - \mathbf{L}) \cdot (F_\gamma - \gamma) \\ &= 2c_n (c'_n + k_n c_{n-1}). \end{aligned}$$

Thus, for  $c_n \neq 0$ ,  $c'_n = \frac{(r_n^2)'}{2c_n} - k_n c_{n-1}$  so that the theorem is proven.  $\square$

**Corollary 9** *i) A point of a null Cartan curve in  $\mathcal{M}^{n+2}$  is a vertex if and only if  $c'_n + k_n c_{n-1} = 0$  at that point.*

*ii) A null Cartan curve in  $\mathcal{M}^{n+2}$  lies on a pseudo-sphere if and only if  $c'_n + k_n c_{n-1} = 0$ .*

**Proof** *i)* Since we have that  $F'_\gamma = (c'_n + k_n c_{n-1}) \mathbf{W}_n$ , it is clear from Lemma 5.

*ii)* Since we know the equality  $c'_n + k_n c_{n-1} = \frac{(r_n^2)'}{2c_n}$  from the preceding theorem,  $c'_n + k_n c_{n-1} = 0$  if and only if  $r_n$  is a constant, which implies that the null Cartan curve lies on the pseudo-sphere.  $\square$

**Remark 10** *The focal curve in  $\mathcal{M}^4$  can be given by  $F_\gamma = \gamma + \frac{1}{k_2} \mathbf{W}_2$ . We conclude from Corollary 9 that a null Cartan curve is a pseudo-spherical in  $\mathcal{M}^4$  if and only if  $k'_2 = 0$ . This characterization corresponds to Theorem 3.2 in [4]. Moreover, Theorem 2 in [15] says that a null Cartan curve is a pseudo-spherical in  $\mathcal{M}^{n+2}$  if and only if  $\sum_{i=2}^n c_i^2 = r^2$  for  $c_n \neq 0$ . This result can be reduced by Corollary 9 to just  $c'_n + k_n c_{n-1} = 0$ , which is the necessary and sufficient condition in order that a null Cartan curve is a pseudo-spherical.*

The following theorem states that the Cartan curvatures can be found by means of the focal curvatures.

**Theorem 11** *The Cartan curvatures  $k_i$ ,  $i = 2, \dots, n$  of a null Cartan curve  $\gamma$  in  $\mathcal{M}^{n+2}$  are expressed by means of the focal curvatures of  $\gamma$  by the formula*

$$k_2 = \frac{1}{c_2}, \quad k_i = \frac{c_2 c'_2 + c_3 c'_3 + \dots + c_{i-1} c'_{i-1}}{c_{i-1} c_i} \quad \text{for } i \geq 3. \tag{10}$$

**Proof** Using Theorem (8), we get that

$$k_2 = \frac{1}{c_2}, k_3 = \frac{c_2c'_2}{c_2c_3} \text{ and } k_4 = \frac{c_2c'_2 + c_3c'_3}{c_3c_4}.$$

Suppose that

$$k_i = \frac{c_2c'_2 + c_3c'_3 + \dots + c_{i-1}c'_{i-1}}{c_{i-1}c_i}.$$

Combining the scalar Frenet equations and this equality, we obtain

$$c_{i+1}k_{i+1} = c'_i + c_{i-1}k_i = c'_i + \frac{c_2c'_2 + c_3c'_3 + \dots + c_{i-1}c'_{i-1}}{c_i}$$

$$k_{i+1} = \frac{c_2c'_2 + c_3c'_3 + \dots + c_i c'_i}{c_i c_{i+1}}$$

so that formula (10) is valid by induction. □

**Definition 12** A point of a null Cartan curve is said to be a pseudo-vertex if the center of the osculating hypersphere at that point lies on the osculating Lorentzian hyperplane at that point (that is, if  $c_n = 0$ ).

**Theorem 13** i) For  $2 \leq l < n$ , the radius  $r_k$  of  $(l+1)$ -osculating pseudo-sphere of a null Cartan curve in  $\mathcal{M}^{n+2}$  is critical at a point if and only if

$$c_3 = 0 \qquad \qquad \qquad \text{for } l = 2,$$

$$\text{either } c_l = 0 \text{ or } c_{l+1} = 0 \qquad \qquad \text{for } 2 < l < n,$$

ii) The radius of osculating pseudo-sphere of a null Cartan curve is critical at a point if and only if such point is either a pseudo-vertex or a vertex.

**Proof** Taking the derivative of radius  $r_l^2 = c_2^2 + c_3^2 + \dots + c_l^2$  of  $(l+1)$ -osculating pseudo-sphere and using formula (10), we get

$$r_l r'_l = c_l c_{l+1} k_{l+1}.$$

For a generic null Cartan curve, the Cartan curvatures do not vanish from [6] (see Proposition 3.3). Thus,  $r'_l$  if and only if  $c_l = 0$  and  $c_{l+1} = 0$ . Furthermore, since the function  $c_2 = 1/k_2$  never vanishes for a smooth null Cartan curve, we conclude  $c_3 = 0$  for  $l = 2$ . Lastly, we know  $(r_n^2)' = 2c_n(c'_n - k_n c_{n-1})$  from Theorem (8) for  $l = n$ , which completes the proof. □

**Remark 14** The results of this section show that the focal curve of a null Cartan curve in  $\mathcal{M}^{n+2}$  has similar properties to the focal curve of a space curve in Euclidean space  $\mathbb{R}^n$ .

**4. Null acceleration directed distance function and acceleration focal curves**

In this section, we introduce a concept of acceleration focal curve for any null Cartan curve  $\gamma$  in  $\mathcal{M}^{n+2}$  by using a family of smooth functions

$$G : I \times \mathcal{M}^{n+2} \rightarrow \mathbb{R}$$

defined by

$$G(t, \mathbf{v}) = (\mathbf{v} - \gamma(t)) \cdot \mathbf{W}_1(t).$$

We call the function  $G$  a *null acceleration directed distance function* of  $\gamma$  in  $\mathcal{M}^{n+2}$ . We denote  $g_{\mathbf{v}_0}(t) = G(t, \mathbf{v}_0)$  for any fixed vector  $\mathbf{v}_0 \in \mathcal{M}^{n+2}$ .

**Definition 15** Let  $\gamma : I \rightarrow \mathcal{M}^{n+2}$  be a null Cartan curve. An acceleration focal curve (A-focal curve)  $\Gamma(t)$  of  $\gamma$  in  $\mathcal{M}^{n+2}$  is the locus of points at which  $\gamma$  and  $g_{\Gamma(t)}^{-1}(0)$  have at least  $(n+2)$ -point contact for all  $t \in I$ .

**Proposition 16** Let  $\gamma : I \rightarrow \mathcal{M}^{n+2}$  be a null Cartan curve. The tangent vector of acceleration focal curve of  $\gamma$  is a linear combination of the null Cartan vectors  $\mathbf{L}$  and  $\mathbf{N}$  of  $\gamma$ .

**Proof** The function  $g_{\Gamma}(t)$  can be written as follows:

$$g_{\Gamma} = \Gamma \cdot \mathbf{W}_1 - h,$$

where  $h = \gamma \cdot \mathbf{W}_1$ . From the definition of A-focal curve, we have

$$\begin{aligned} \gamma'' \cdot \Gamma(t) - h &= 0, \\ \gamma''' \cdot \Gamma(t) - h' &= 0, \\ &\vdots \\ \gamma^{(n+3)} \cdot \Gamma(t) - h^{(n+1)} &= 0. \end{aligned} \tag{11}$$

Taking the derivative of these equations, we obtain

$$\begin{aligned} \gamma'' \cdot \Gamma'(t) + \gamma''' \cdot \Gamma(t) - h' &= 0, \\ \gamma''' \cdot \Gamma'(t) + \gamma^{(4)} \cdot \Gamma(t) - h'' &= 0, \\ &\vdots \\ \gamma^{(n+2)} \cdot \Gamma'(t) + \gamma^{(n+3)} \cdot \Gamma(t) - h^{(n+2)} &= 0, \end{aligned} \tag{12}$$

respectively. Combining the  $(i+1)^{th}$  equation of system (11) with the  $i^{th}$  equation of system (12), we conclude that

$$\begin{aligned} \gamma'' \cdot \Gamma'(t) &= 0, \\ \gamma''' \cdot \Gamma'(t) &= 0, \\ &\vdots \\ \gamma^{(n+2)} \cdot \Gamma'(t) &= 0. \end{aligned} \tag{13}$$



System (13) implies that  $\Gamma(t)$  is a linear combination of the null Cartan vectors  $\mathbf{L}$  and  $\mathbf{N}$  of  $\gamma$ . □

**Remark 17** We can state the acceleration focal curve  $\Gamma$  of the null Cartan curve  $\gamma$  as

$$\Gamma = \gamma + a\mathbf{L} + b\mathbf{N} + d_2\mathbf{W}_2 + d_3\mathbf{W}_3 + \cdots + d_n\mathbf{W}_n, \tag{14}$$

where the coefficients  $a, b, d_2, \dots, d_n$  are smooth functions of pseudo-arc parameter of  $\gamma$ . We call these coefficients acceleration focal curvatures (A-focal curvatures) of a null Cartan curve. Unlike the focal curve defined by (5), the A-focal curve parametrized by (14) can be determined in the 3-dimensional Minkowski space.

**Theorem 18** The A-focal curvatures of a null Cartan curve  $\gamma$  satisfy the following equations:

$$\begin{pmatrix} a \\ d'_2 \\ d'_3 \\ d'_4 \\ \vdots \\ d'_{n-2} \\ d'_{n-1} \\ d'_n \end{pmatrix} = \begin{pmatrix} -k_1 & 0 & \cdots & & 0 & 0 & 0 \\ -k_2 & 0 & k_3 & \cdots & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_4 & \cdots & & \vdots \\ 0 & 0 & -k_4 & & & & \vdots \\ 0 & & & & & & \vdots \\ & & & & 0 & k_{n-1} & 0 \\ & & & & -k_{n-1} & 0 & k_n \\ 0 & & & \cdots & 0 & -k_n & 0 \end{pmatrix} \begin{pmatrix} b \\ d_2 \\ d_3 \\ d_4 \\ \vdots \\ d_{n-2} \\ d_{n-1} \\ d_n \end{pmatrix}. \tag{15}$$

**Proof** Deriving the null focal curve  $\Gamma$  defined by (14) with respect to pseudo-arc length parameter of  $\gamma$  and using equations (3), we get

$$\begin{aligned} \Gamma' &= (1 + a' - k_2d_2)\mathbf{L} + b'\mathbf{N} + (a + k_1b)\mathbf{W}_1 + (d'_2 + k_2b - k_3d_3)\mathbf{W}_2 + (d'_3 + k_3d_2 - k_4d_4)\mathbf{W}_3 \\ &+ \cdots + (d'_{n-1} + k_{n-1}d_{n-2} - k_nd_n)\mathbf{W}_{n-1} + (d'_n + k_nd_{n-1})\mathbf{W}_n. \end{aligned}$$

From proposition 16, we can say that  $\Gamma' = (1 + a' - k_2d_2)\mathbf{L} + b'\mathbf{N}$  and

$$\begin{aligned} a &= -k_1b \\ d'_2 &= -k_2b + k_3d_3 \\ d'_3 &= -k_3d_2 + k_4d_4 \\ &\vdots \\ d'_{n-1} &= -k_{n-1}d_{n-2} + k_nd_n \\ d'_n &= -k_nd_{n-1} \end{aligned}$$

□

**Corollary 19** If the null Cartan curve  $\gamma$  is a pseudo-spherical if and only if  $d_{n-1} = c_{n-1}$  and  $d'_n = c'_n$ .

The above Corollary 19 introduces an alternative way to learn whether a null Cartan curve is a pseudo-spherical. Now let us define the notion of a null vertex point of null Cartan curve.

**Definition 20** A null vertex point of null Cartan curve is a point at which the velocity vector of the A-focal curve is proportional to the null transversal vector.

From the above definition, we can say that if all points of null Cartan curve  $\gamma$  are null vertex, the A-focal curve is a null curve. Moreover, we have the following result.

**Theorem 21** All points of null Cartan curve  $\gamma$  are null vertex if and only if the Cartan curvatures  $k_i$ ,  $i = 1, 2, \dots, n$  in  $\mathcal{M}^{n+2}$  are given by the A-focal curvatures of  $\gamma$  as follows:

$$k_1 = \frac{-b}{a}, \quad k_2 = \frac{a' + 1}{d_2}$$

$$k_i = \frac{(1 + a')b + d_2d'_2 + d_3d'_3 + \dots + d_{i-1}d'_{i-1}}{d_{i-1}d_i} \quad \text{for } i \geq 3. \tag{16}$$

**Proof** Since we have  $\Gamma' = (1 + a' - k_2d_2)\mathbf{L} + b'\mathbf{N}$ , the definition of null vertex point gives the equation

$$k_2 = \frac{a' + 1}{d_2}.$$

From Theorem 18, we know the equation  $k_1 = \frac{-b}{a}$  and we can find

$$d'_2 = -\frac{a' + 1}{d_2}b + k_3d_3 \Rightarrow$$

$$k_3 = \frac{(1 + a')b + d_2d'_2}{d_2d_3}.$$

The other formulas in (16) are obtained by induction and using equations (15). □

Let us find the parameter of the A-focal curve in  $\mathcal{M}^3$ . Using definition (15), the A-focal curve  $\Gamma_3$  of a null Cartan curve  $\gamma$  is given by

$$\Gamma_3(t) = \gamma(t) - \frac{k}{k'}\mathbf{L} + \frac{1}{k'}\mathbf{N},$$

where  $k$  is the Cartan curvature of  $\gamma$ . The A-focal curve  $\Gamma_3$  is a spacelike curve if  $k < 0$  and a timelike curve if  $k > 0$  for all  $t$ .

**Lemma 22** The curvatures and torsions of timelike and spacelike A-focal curves  $\Gamma_3$  obtained by the null Cartan curve  $\gamma$  in  $\mathcal{M}^3$  are given by

$$\kappa_{\Gamma_3} = \frac{(k')^3}{k''(2k)^{3/2}}, \quad \tau_{\Gamma_3} = \sqrt{2k} \tag{17}$$

and

$$\kappa_{\Gamma_3} = \frac{(k')^3}{k''(-2k)^{3/2}}, \quad \tau_{\Gamma_3} = \sqrt{-2k}, \tag{18}$$

respectively, where  $k(t)$  is the curvature function of  $\gamma$  with the pseudo-arc parameter  $t$ .

**Proof** We can find the arc-length element  $ds$  of  $\Gamma_3$  as  $ds = \frac{(k')^2 \sqrt{|2k|}}{k''}$ . The Frenet vectors of timelike A-focal curve can be calculated as the following:

$$\begin{aligned} \mathbf{t}_{\Gamma_3} &= \frac{1}{\sqrt{2k}} (k\mathbf{L} - \mathbf{N}), \\ \mathbf{n}_{\Gamma_3} &= \frac{1}{\sqrt{2k}} (k\mathbf{L} + \mathbf{N}), \\ \mathbf{b}_{\Gamma_3} &= \mathbf{W}, \end{aligned}$$

where  $\{\mathbf{L}, \mathbf{N}, \mathbf{W}\}$  is a null Cartan frame of  $\gamma$  and  $\mathbf{t}_{\Gamma_3}$ ,  $\mathbf{n}_{\Gamma_3}$ , and  $\mathbf{b}_{\Gamma_3}$  are a unit tangent vector, normal vector, and binormal vector, respectively. Using the Frenet–Serret equations for non-lightlike curve  $\alpha$  in  $\mathcal{M}^3$  stated by

$$\frac{d}{ds} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ \varepsilon_{\mathbf{b}}\kappa_\alpha & 0 & \tau_\alpha \\ 0 & \varepsilon_{\mathbf{t}}\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}$$

where  $\varepsilon_{\mathbf{b}} = \varepsilon_{\mathbf{t}} = \pm 1$  and , we can obtain equations (17). We can similarly find the equations (18) for the spacelike A-focal curve. □

**Corollary 23** *A null Cartan curve in  $\mathcal{M}^3$  has no null vertex point.*

Now we investigate the A-focal curve  $\Gamma_4$  in Minkowski space-time. From definition 15, the following equations are satisfied:

$$\begin{aligned} a + k_1b &= 0, \\ k'_1b + k_2d_2 &= 0, \\ (k''_1 - k_2^2)b + k'_2d_2 &= 0. \end{aligned} \tag{19}$$

The last two equations imply the equation

$$k'_1k'_2 - k''_1k_2 + k_2^3 = 0.$$

Moreover, the tangent vector of the A-focal curve can be found as

$$\Gamma'_4 = (1 + a' - k_2d_2)\mathbf{L} + b'\mathbf{N} + d'_2\mathbf{W}_2.$$

Proposition 16 says the equality  $d'_2 = 0$ . Using equations (19) and  $d'_2 = 0$ , the parametrization of the A-focal curve can be found as follows:

$$\Gamma_4 = \gamma + e^{-\int \frac{k_2^2}{k_1} dt} \left( -k_1\mathbf{L} + \mathbf{N} - \frac{k'_1}{k_2}\mathbf{W}_2 \right).$$

We have the following Lemma from the above calculations.

**Lemma 24** *Let  $\gamma : I \rightarrow \mathcal{M}^4$  be a null Cartan curve. If  $\gamma$  and  $g_{\mathbf{v}_0}^{-1}(0)$  have at least 4-point contact in  $\mathcal{M}^4$ , then the equation  $k'_1k'_2 - k''_1k_2 + k_2^3 = 0$  is satisfied.*

### 5. Concluding remarks

In this paper, the geometry of a focal curve, defined as the locus of centers of osculating hyperspheres of a null Cartan curve, was studied. Moreover, the notion of an A-focal curve (acceleration focal curve) for a null Cartan curve was represented by using the null acceleration directed distance function in  $\mathcal{M}^{n+2}$ .

Using the volumelike distance function, the paper [19] investigated the singularities of the surface  $FS$ , defined by (1), by means of the singularity theory in Minkowski 3-space. Similarly, a surface associated with a null Cartan curve  $\gamma$  in  $\mathcal{M}^3$  will be defined as follows:

$$FW(s, \mu) = \gamma(s) + \mu(k(s)\mathbf{L}(s) - \mathbf{N}(s))$$

or

$$FW(s, \mu) = \gamma(s) + \mu\left(-\mathbf{L}(s) + \frac{1}{k(s)}\mathbf{N}(s)\right),$$

which is the discriminant set of the function germ  $G : I \times \mathcal{M}^3 \rightarrow \mathbb{R}$ , in the next study. The surface  $FW(s, \mu)$  will be called *acceleration focal surface* of a null Cartan curve  $\gamma$  and the singularities of this surface  $FW(s, \mu)$  will be examined by using the null acceleration directed distance function in singularity theory.

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