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FREE BOUNDARY VALUE PROBLEMS FOR ABSTRACT ELLIPTIC EQUATIONS AND APPLICATIONS

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ABSTRACT. Free boundary value problem for abstract elliptic equations with variable coefficients is studied. The equations involve linear operators in Banach space E. The uniform maximal regularity properties and Fredholmness of this problem are obtained in E-valued Hölder spaces. It is proven that the corresponding differential operator is positive and is a generator of an analytic semigroup. In application, the maximal regularity properties of Cauchy problem for abstract parabolic equation and anisotropic elliptic equations are established.

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1. INTRODUCTION, NOTATIONS AND BACKGROUND

The maximal regularity properties of boundary value problems (BVPs) for differential-operator equations (DOEs) have found many applications in PDE, pseudo-differential equations arising in various physical processes (see for example [1, 3–6, 8, 10, 12, 13, 15–21, 23, 25, 26]). Regularity properties of free BVPs for elliptic and parabolic equations are studied for example in [2, 7, 9, 11, 14, 22].

Let Ω be a domain in $\mathbb{R}^n$ and E be a Banach space. $C_b^{(m)}(\Omega;\, E)$ will denote the spaces of all E-valued bounded uniformly stongly continuous and m-times continuously differentiable functions on Ω. For $m = 0$ this space is denoted by $C_b(\Omega;\, E)$. Let $\mathbb{C}$ denote the set of complex numbers. For $E = \mathbb{C}$ the space $C^{(m)}(\Omega;\, E)$ will be denoted by $C_b^{(m)}(\Omega)$. Moreover, $C_b^{\infty}(\Omega;\, E)$ denotes the space of all E-valued bounded strongly continuously differentiable functions of arbitrary order. We put $\mathbb{R} =(-\infty,\, \infty)$ and $\mathbb{R}_+ = (0,\, \infty)$. Let $f(x)$ be a $E-$valued function and $f(x) \neq 0$. Consider

$$\Omega_f = \{(x,\, y) \in \mathbb{R} \times \mathbb{R}_+,\, f \in C_b(\mathbb{R};\, E),\, 0 < y < \|f(x)\|\}.$$

The boundaries of Ω_f are given by

$$\Gamma_0 = \mathbb{R} \times 0, \quad \Gamma_f = \{(x,\, y) \in \mathbb{R} \times \mathbb{R}_+,\, y = \|f(x)\|\}.$$

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Consider the following problem: Given $f_0,\, \nu \in C_b^{(2)}(\mathbb{R};\, E)$. Find a pair of functions $(u,\, f)$ possessing the regularity

$$\begin{gathered} f \in C_b^{(1)}([0,\, T);\, C_b^{(1)}(\mathbb{R};\, E)), \\ u(t,\, .) \in W_p^2(\Omega_{f(t)};\, E), \quad t \in J = [0,\, T), \end{gathered} \tag{1.1}$$

and satisfying the following equations a.e.

$$\begin{gathered} -\Delta u(t,\, z) + A(x)u(t,\, z) = 0, \quad t \in J,\, z \in \Omega_{f(t)}, \\ \frac{\partial u}{\partial y} = 0, \quad t \in J,\, z \in \Omega_{f(t)}, \\ u(t,\, z) = f(t,\, x), \quad t \in J,\, z \in \Gamma_{f(t)}, \\ \lim_{z \to \infty} u(t,\, z) = \nu(t), \quad t \in [0,\, T), \\ f_t(t,\, x) + \sqrt{1 + f_x^2(t,\, x)}\frac{\partial}{\partial n} u(t,\, z) = 0, \quad t \in (0,\, T),\, z \in \Gamma_{f(t)}, \\ f(0,\, x) = f_0(x), \quad x \in \mathbb{R}, \end{gathered} \tag{1.2}$$

where A is a linear operator in a Banach space E and $z = (x,\, y)$ represents a generic point in $\bar{\Omega}_f$. Moreover, Δ denotes the Laplace operator with respect to the Euclidean metric, $\frac{\partial}{\partial n}$ denotes the derivative in the direction of the outer unit normal n at $\Gamma_{f(t)}$. Maximal regularity properties of partial DOEs in L_p spaces were studied in [3, 5, 8, 17–21, 23]. The results in [5] and [17–21, 23] were restricted to rectangular domains and equations that did not contain mixed derivatives in leading parts. Moreover, the problems investigated in [3] and [6] involve only bounded operator coefficients. In [23] the Dirichlet problem for the elliptic differential-operator equation of the second order in general domains was studied. In contrast to all above, we study general BVP (1.1) for equation with unbounded operator coefficients in general domains. Consider the BVP

$$\begin{gathered} Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} - A(x)u(x) = F(x), \\ L_1 u = u|_{\Gamma} = 0, \end{gathered} \tag{1.3}$$

where Γ is the boundary of region $G \subset \mathbb{R}^n$ and a_{ij} are real-valued functions on $\bar{G}$.

We say that the problem (1.3) is maximal H-regular (or separable in Hölder space C^γ) if:

(1) for all $F \in C^\gamma(G;\, E)$ there exists a unique solution $u \in C^{2,\gamma}(G;\, E(A),\, E)$ satisfying (1.3) a.e. on G;
(2) there exists a positive constant C independent of F such that

$$\sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{C^\gamma(G;E)} + \|Au\|_{C^\gamma(G;E)} \leqslant C\|F\|_{C^\gamma(G;E)}.$$

Let G denote the operator generated by the problem (1.3) for $\lambda = 0$, i.e.,

$$\begin{gathered} D(G) = C_0^{2,\gamma}(G;\, E(A),\, E) = \{u \in C^{2,\gamma}(G;\, E) \cap C(G;\, E(A)),\, u|_{\Gamma} = 0\}, \\ Gu = Lu. \end{gathered}$$

Section 1 collects definitions and background materials, embedding theorems of Sobolev–Lions spaces. Let $\mathbb{C}$ be the set of complex numbers and

$$S(\varphi) = \{\lambda \in \mathbb{C},\ |\arg \lambda| \leqslant \varphi\} \cup \{0\}, \quad 0 \leqslant \varphi < \pi.$$

Let E_1 and E_2 be two Banach spaces. $L(E_1,\ E_2)$ denotes the space of all bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ it will be denoted by $L(E)$. A linear operator A is said to be φ-positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and

$$\|(A + \lambda I)^{-1}\|_{L(E)} \leqslant M(1 + |\lambda|)^{-1}$$

with $\lambda \in S(\varphi)$, $\varphi \in [0,\ \pi)$, where I is an identity operator in E.

Sometimes $A + \lambda I$ will be written as $A + \lambda$ and will be denoted by A_λ. It is known [24, §1.15.1] that there exist fractional powers A^θ of positive operator A. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with graphical norm

$$\|u\|_{E(A^\theta)} = (\|u\|^p + \|A^\theta u\|^p)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,\ -\infty < \theta < \infty.$$

A linear operator $A(x)$ is said to be positive in E uniformly in x if $D(A(x))$ is independent of x, $D(A(x))$ is dense in E and

$$\|(A(x) + \lambda I)^{-1}\| \leqslant M(1 + |\lambda|)^{-1}$$

for all $\lambda \in S(\varphi)$ and $\varphi \in [0,\ \pi)$.

Let Ω be a domain in $\mathbb{R}^n$. $C(\Omega,\ E)$ and $C^m(\Omega;\ E)$ will denote the spaces of all E-valued bounded uniformly strongly continuous and m-times continuously differentiable functions on Ω, respectively.

Let $0 < \gamma \leqslant 1$. $C^\gamma(\Omega;\ E)$ denotes the space of all E-valued strongly bounded continuous functions that are defined on $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_{C^\gamma(\Omega;E)} = \|f\|_{C(\Omega;E)} + [f]^\gamma(E),$$

where

$$[f]^\gamma(E) = \sup_{x \neq y, x, y \in \Omega} \frac{\|f(x) - f(y)\|_E}{|x - y|^\gamma}.$$

$C^{\gamma,m}(\Omega;\ E)$ denotes the space of all E-valued strongly bounded continuous functions that are defined on $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_{C^{\gamma,m}(\Omega;E)} = \|f\|_{C^m(\Omega;E)} + \|f^{(m)}\|_{C^\gamma(\Omega;E)} < \infty,$$
$$C^{\gamma,m}(\Omega;\ E) = \{f \in C^m(\Omega;\ E),\ f^{(m)} \in C^\gamma(\Omega;\ E)\},$$

Let E_0 and E be two Banach spaces such that E_0 is continuously and densely embedded into E. Let m be a natural number. Let $\alpha = (\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n)$ be n tuples of nonnegative integer numbers and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

$C^{m,\gamma}(\Omega;\ E_0,\ E)$ denotes the space of all E_0-valued bounded uniformly stongly continuous and m-times continuously differentiable functions on Ω with norm

$$\|f\|_{C^{m,\gamma}(\Omega;E_0,E)} = \|f\|_{C^{m,\gamma}(\Omega;E)} + \|f\|_{C^\gamma(\Omega;E_0)}.$$

For $E_0 = E$ the space $C^{m,\gamma}(\Omega;\, E_0,\, E)$ is denoted by $C^{m,\gamma}(\Omega;\, E)$. Let A be a linear operator in a Banach space E and A be a generator of an analytic semigroup $U(t) = U_A(t)$. Let

$$\begin{aligned} D_A(\theta,\, p) &= \{u \in E,\ \|u\|_{D_A(\theta,p)} = \|t^{1-\theta-\frac{1}{p}} AU(t)u\|_{L^p(0,1;E)} < \infty\} \\ &\qquad \text{for } 1 \leqslant p < \infty \text{ and } \theta \in (0,\, 1), \\ D_A(\theta,\, \infty) &= \{u \in E,\ \|u\|_{D_A(\theta,p)} = \|t^{1-\theta} AU(t)u\|_{L^\infty(0,1;E)} < \infty\} \\ &\qquad \text{for } p = \infty \text{ and } 0 < \theta \leqslant 1; \\ D_A(\theta) &= \{u \in D_A(\theta,\, \infty),\ \lim_{t\to 0} t^{1-\theta} AU(t)u = 0\}. \end{aligned}$$

From [16, Proposition 2.2.2], we obtain the following

$$\begin{aligned} D_A(\theta,\, p) &= (E,\, D(A))_{\theta,p} && \text{for } 1 \leqslant p < \infty,\, \theta \in (0,\, 1), \\ D_A(\theta,\, \infty) &= (E,\, D(A))_{\theta,\infty} && \text{for } 0 < \theta \leqslant 1, \\ D_A(\theta) &= (E,\, D(A))_{\theta}, && \text{for } \theta \in (0,\, 1). \end{aligned}$$

$H(E_0,\, E)$ denotes the class of linear operators that are isomorphsms from E_0 onto E and are negative generators of strongly continuous and analytic semigroups.

Let $S'(\mathbb{R}^n;\, E)$ denote the space of all continuous linear operators $L\colon S(\mathbb{R}^n;\, E) \to E$, equipped with the bounded convergence topology. Recall $S(\mathbb{R}^n;\, E)$ is norm dense in $L^p(\mathbb{R}^n;\, E)$ when $1 \leqslant p < \infty$.

Let $L_q^*(E)$ denote the space of all E-valued functions $u(t)$ such that

$$\|u\|_{L_q^*(E)} = \left(\int_0^\infty \|u(t)\|_E^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty, \quad 1 \leqslant q < \infty,\ \|u\|_{L_\infty^*(E)} = \sup_{0<t<\infty} \|u(t)\|_E.$$

Let F denote the Fourier transform. The E-valued Besov space on $\mathbb{R}^n$ is defined as in [25, §3], i.e.,

$$\begin{aligned} B^s_{p,q}(\mathbb{R}^n;\, E) = \Bigg\{ & u \in S'(\mathbb{R}^n;\, E), \\ & \|u\|_{B^s_{p,q}(\mathbb{R}^n;E)} = \left\| F^{-1} \sum_{k=1}^n t^{\varkappa_k - s_k} (1 + |\xi_k|^{\varkappa_k}) e^{-t|\xi|^2} Fu \right\|_{L_q^*(L_p(\mathbb{R}^n;E))}, \\ & p,\, q \in [1,\, \infty],\ \varkappa_k > s_k,\ s = (s_1,\, s_2,\, \ldots,\, s_n) \Bigg\}. \end{aligned}$$

For appropriate domain $\Omega \subset \mathbb{R}^n$ the space $B^s_{p,q}(\Omega;\, E)$ is defined as usual restriction of the space $B^s_{p,q}(\mathbb{R}^n;\, E)$.

For $E = \mathbb{C}$ the space $B^s_{p,q}(\Omega;\, \mathbb{C})$ will be denoted by $B^s_{p,q}(\Omega)$.

Let $h^s = h^s(\mathbb{R}^n;\, E)$ denote the closure of $S(\mathbb{R}^n;\, E)$ in $B^s_{\infty,\infty}(\mathbb{R}^n;\, E)$. Assume that Ω is an open subset of $\mathbb{R}^n$ and let r_Ω denote the restriction operator with respect to Ω, i.e., $r_\Omega u = u\,|_\Omega$ for $B^s_{\infty,\infty}(\mathbb{R}^n;\, E)$. Here, $h^s(\Omega;\, E)$ is defined as the closure of $r_\Omega(S(\mathbb{R}^n;\, E))$ in $B^\gamma_{\infty,\infty}(\Omega;\, E)$ and $h^{m,\gamma}(\Omega;\, E)$ is defined as the closure of $r_\Omega(S(\Omega;\, E))$ in $C^{m,\gamma}(\Omega;\, E)$. For $E = \mathbb{C}$ the spaces $h^s(\Omega;\, E)$, $h^{m,\gamma}(\Omega;\, E)$ will be denoted by h^s and $h^{m,\gamma}$, respectively. Moreover, let $C_0^s(\Omega;\, E)$ denote the closure of $C^\infty(\Omega;\, E)$ in $C^s(\Omega;\, E)$.

Let

$$h_+^s = h_+^s(\mathbb{R};\, E) = \{g \in h^s,\, g(x) \neq 0\},$$
$$X = h^\alpha(Q;\, E), \qquad X_k = h^{k,\alpha}(Q;\, E),$$
$$Y = h^{2,\alpha}(Q;\, E(A),\, E) = \{u \in h^\alpha(Q;\, D(A)) \cap h^{2,\alpha}(Q;\, E)\}$$

with the norm

$$\|u\|_Y = \|u\|_{h^\alpha(Q;D(A))} + \|u^{(2)}\|_{h^\alpha(Q;E)} < \infty.$$

Here,

$$B_+^{2+\alpha} = \{u \in B_{\infty,\infty}^{2+\alpha}(\mathbb{R}^n;\, E),\, g(x) \neq 0\}$$

and

$$h^{m,\alpha}(A) = h^{m,\alpha}(\mathbb{R};\, D_A(\alpha,\, \infty)), \qquad \alpha \in (0,\, 1).$$

Remark 1.1. In order to formulate our result, let

$$h_\nu^s = \{\nu + g;\, g \in h^s(\mathbb{R};\, E)\}, \qquad h_\nu^{s,\alpha} = \{\nu + g;\, g \in h^{s,\alpha}(\mathbb{R};\, E)\}.$$

Assume that $f \in h_\nu^{2,\alpha}$ given. Let u_f denote the unique solution of the BVP

$$-\Delta u + Au = 0, \qquad \partial_y u = 0 \text{ on } \Gamma_0, \qquad u = f \text{ on } \Gamma_f,$$

where A is a linear operator in a Banach space E. Let

$$k_f = \frac{\|f\|^2}{(1 + \|f\| + \|f_x\|^2)(1 + \|f_x\|^2)}$$

and define

$$V_\nu = \{f \in C_b^2(\mathbb{R};\, E)\}, \qquad f(x) \neq 0, \qquad \partial_y u_f(x,\, f(x)) < k_f \text{ for } x \in \mathbb{R}.$$

It is clear that $u_\nu \equiv \nu$. Hence, $\nu \in V_\nu$. More precisely, by following [Lemma 5.10] it can be shown that V_ν is a open neighborhood of ν in $h_\nu^{s,2}(E)$ and that

$$\operatorname{diam}_{h^{2,s}}(V_\nu) = \sup_{g,\upsilon \in V_\nu} \|g - \upsilon\|_{h^{s,2}} = \infty.$$

Suppose now that $(u,\, f)$ is a classical solution of (1.1)–(1.2). We call $(u,\, f)$ a classical Hölder solution on J if it possesses the additional regularity

$$f \in C(J;\, V_\nu) \cap C^1(J;\, h_\nu^{1,\alpha}), \qquad u(t,\, .) \in h_\nu^{2,\alpha}(\mathbb{R};\, E), \qquad t \in J.$$

We will prove the following main result

Theorem 1. *Given $f_0 \in V_\nu$, there exist $t^+ = t^+(f_0)$ and a unique maximal classical Hölder solution $(u,\, f)$ of problem (1.1)–(1.2) on $[0,\, t^+)$. Moreover, the mapping $(t,\, f_0) \to f$ defines a local C^∞-semiflow on V_ν. If $t^+ < \infty$ and $f\colon [0,\, t^+) \to V_\nu$ is uniformly continuous then either*

$$\lim_{t\to t^+} \|f(t,\, .)\|_{h^{2,\alpha}} = \infty \quad \text{or} \quad \lim_{t\to t^+} \inf_{\upsilon \in V_\nu} \|f(t,\, .) - \upsilon\|_{h^{2,\alpha}} = 0.$$

In the first stage, we transform problem (1.1)–(1.2) into a nonlinear problem on a fixed domain

$$\frac{df}{dt} + O(f) = 0, \qquad f(0) = f_0 \tag{1.4}$$

with respect to only the unknown function f, which determines the free boundary Γ_f, where O is a nonlinear operator in E. By using the solution of the problem (1.4), we will show the exsistence of regular solution of the free BVP (1.1)–(1.2).

2. Transformed Problem

Let $\nu = \nu(t) > 0$ be fixed. Define

$$G_\nu = \{g \in C_b^{(2)}(\mathbb{R};\, E),\, \nu(t) + g(x) \neq 0\}.$$

Consider the following transformation

$$(x,\, y) = \varphi(x',\, y') = \varphi_g(x',\, y') = \left(x',\, 1 - \frac{y'}{\nu + g(x')}\right), \quad \text{for } (x',\, y') \in \Omega_f. \tag{2.1}$$

It is easily verified that φ_g is a diffeomorphism of class C^2 which maps Ω_g onto the strip $Q = \mathbb{R} \times (0,\, 1)$. Moreover,

$$\begin{aligned} (x',\, y') = \varphi^{-1}(x,\, y) = \varphi_g^{-1}(x,\, y) = (x,\, (1-y)g(x)) \quad \text{for } (x,\, y) \in Q. \\ \varphi_* u = \varphi_*^g u = u(\varphi_g^{-1}(x,\, y)) \quad \text{for } u \in W_p^2(\Omega_g;\, E(A),\, E). \end{aligned} \tag{2.2}$$

Let u be an E-valued function defined on Q. Here, $u_{|\Gamma_i}$ denotes the restriction of u on Γ_i, where

$$\Gamma_i = \mathbb{R} \times \{i\}, \quad \gamma_i u = u_{|\Gamma}, \quad i = 0,\, 1.$$

Lemma 2.1. *Given $g \in \Phi_\eta$ and $\upsilon \in C^{2,\gamma}(Q;\, E(A),\, E)$. Under the transformation (2.2) the operators in (1.1) are transformed into the following:*

$$\begin{aligned} B(g)\upsilon &= -\varphi^g * (\Delta + A)(\varphi_g * \upsilon) \quad && \text{on } [0,\, T) \times Q, \\ B_i(g)\upsilon &= \varphi_*^g((\nabla(\varphi_g^*),\, n_i)) && \text{on } (0,\, T) \times \Gamma_i, \end{aligned} \tag{2.3}$$

where $n_0 = (-g_x,\, 1)$, $n_1 = (0,\, -1)$ denote the outer normals according to Γ_f and Γ_0, i.e.,

$$\begin{aligned} B_0(g)\upsilon &= \varphi_*^g\left(-\frac{\partial}{\partial x}\varphi_g^*\upsilon + \frac{\partial}{\partial y}\varphi_g^*\upsilon\right) \quad && \text{on } (0,\, T) \times \Gamma_0, \\ B_1(f)\upsilon &= \varphi_*^g(-\frac{\partial}{\partial y}\varphi_g^*\upsilon) && \text{on } [0,\, T) \times \Gamma_1. \end{aligned} \tag{2.4}$$

Lemma 2.2. *Given $g \in \Phi_\nu$ and $\upsilon \in C^{2,\gamma}(Q;\, E(A),\, E)$. Under the transformation (2.2) the problem (1.1) is transformed into the following:*

$$\begin{gathered} B(f)\upsilon = -\varphi^f * (\Delta + A)(\varphi_g * \upsilon) = 0 \quad \text{on } [0,\, T) \times Q, \\ \upsilon = f \quad \text{on } [0,\, T) \times \Gamma_0, \\ B_1(f)\upsilon = 0 \quad \text{on } [0,\, T) \times \Gamma_1, \\ \lim_{z\to\infty} \upsilon(t,\, z) = 0 \quad \text{on } [0,\, T), \\ \frac{\partial g}{\partial t} + B_0(g)\upsilon = 0 \quad \text{on } (0,\, T) \times \Gamma_0, \\ g(0,\, .) = g_0(x) \quad \text{on } \mathbb{R}, \end{gathered} \tag{2.5}$$

A pair $(\upsilon,\, g)$ is called a solution of the problem (2.5) if

$$\begin{gathered} g \in C_b([0,\, T);\, \Phi) \cap C_b^{(1)}([0,\, T);\, C_b^{(1)}(\mathbb{R})), \\ u(t,\, .) \in W_p^2(\Omega_{f(t)};\, E), \quad t \in [0,\, T) \end{gathered} \tag{2.5.0}$$

and $(\upsilon,\, g)$ satisfies (2.5) a.e. on $[0,\, T) \times Q$.

Condition 2.1. Assume the following conditions are satisfied:

(1) $\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geqslant C|\xi|^2$, for $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ and $C > 0$;

(2) operator A is a φ-positive operator in a Banach space E for some $\varphi \in (0, \pi]$.

In a similar way as in [9, Lemma 2.2], we obtain

Lemma 2.3. *Assume that Condition 2.1 is satisfied. Then for given $g \in \Phi$, we have*

$$\begin{aligned} B(g)u &= \sum_{i,j=1}^{2} -a_{ij}(g)\frac{\partial^2 u}{\partial x_i \partial x_j} + a_2(g)\frac{\partial u}{\partial x_2} + A_g u, \\ B_i(g)u &= \sum_{j=1}^{2} b_{ji}(g)\gamma_i \frac{\partial u}{\partial x_j}, \qquad i = 0,\, 1, \end{aligned} \tag{2.6}$$

and

$$\sum_{i,j=1}^{2} a_{ij}(g)\xi_i\xi_j \geqslant \alpha(g)|\xi|^2, \qquad \textit{for } \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n,$$

where $\alpha(g) > 0$, γ_i are trace operators from Q to Γ_i, $i = 0, 1$,

$$a_{11}(g) = 1, \quad a_{12}(g) = a_{21}(g) = \frac{\beta g_{x_1}}{\nu + g}, \quad a_{22}(g) = \frac{1 + \beta^2 g_{x_1}^2}{(\nu + g)^2}, \quad \beta = 1 - x_1,$$

$$a_2(g) = \frac{\beta}{\nu + g}\left[\frac{2g_{x_1}^2}{\nu + g} - g_{x_1x_1}\right], \quad b_{10}(g) = -g_{x_1}, \quad b_{20}(g) = -\frac{1 + g_{x_1}^2}{\nu + g}, \tag{2.7}$$

$$b_{11}(g) = 0, \quad b_{21}(g) = \frac{1}{\nu + g}, \quad \alpha(g) = \frac{1}{1 + (\nu + g)^2 + \beta^2 g_{x_1}^2}, \quad A(g) = A(\varphi_g).$$

3. Abstract Elliptic Equation in the Fixed Domain

In this section we study the elliptic BVP

$$B(g)u = -\sum_{i,j=1}^{2} a_{ij}(g)\frac{\partial^2 u}{\partial x_i \partial x_j} + a_2(g)\frac{\partial u}{\partial x_2} + A(x)u = f, \tag{3.1}$$

$$B_i(g)u = \sum_{j=1}^{2} b_{ji}(g)\gamma_i \frac{\partial u}{\partial x_j} = f_i, \qquad i = 0,\, 1, \tag{3.2}$$

where $B(g)$ and $B_i(g)$ are differential operators defined by (2.6).

We will derive a priori estimates as well as isomorphism properties in framework of abstract Hölder spaces.

Condition 3.1. Assume that the following conditions are satisfied:

(1) $a_{ij} \in C^{0,\alpha}(\bar{G})$, $a_{ij} = a_{ji}$;

(2) $\sum_{i,j=1}^{2} a_{ij}(g)\xi_i\xi_j \geqslant C(g)|\xi|^2$, for $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ and $C(g) > 0$;

(3) operator $A(g)$ is uniformly positive in a Banach algebra E for some $\varphi \in (0, \pi]$.

Below, $\partial B(g)[\psi, .v]$ denotes the Gateaux derivative of operator function $B(g)$ at ψ in the direction of v.

Lemma 3.1. *Suppose Condition* 3.1 *is satisfied and* $A(g)$ *is Gateaux differentiable for* $g \in h_{\Phi}^{2,\alpha}$, $\alpha \in (0, 1)$. *Then the transformation* $g \to O_0(g) = \{B(g), B_0(g)\}$ *is a bounded linear operator-function from* Y *into* $X \times h^{1,\alpha}(A)$ *and has continuous derivatives of any order with respect to* $g \in h_{\Phi}^{2,\alpha}$, *i.e.*

$$B(.) \in C^{\infty}(h_{\Phi}^{2,\alpha};\, L(Y,\, X)), \quad B_0(.) \in C^{\infty}(h_{\Phi}^{2,\alpha};\, L(Y,\, E))$$

and

$$\partial B(g)[u,\, \upsilon] = (\partial B(g)u)\upsilon = \frac{2\beta}{\nu+g}\left\{\left(\frac{g_x u}{\nu+g} - u_x\right)\upsilon_{x_1x_2} + \frac{u}{(\nu+g)^2}\left(\left(\frac{1}{\beta}+\beta g_x^2\right) - \frac{\beta}{\nu+g} g_x u_x\right)\upsilon_{x_2x_2} - \left(\frac{g_x u}{(\nu+g)^2} - \frac{g_{xx}u + 4g_x u_x}{2(\nu+g)} + \frac{u_{xx}}{2}\right)\upsilon_{x_2}\right\} + \partial A(g)[u,\, \upsilon],$$

$$\partial B_0(g)[u,\, \upsilon] = -u_x\upsilon_{x_1} + \frac{\beta}{\nu+g}\left[\frac{u(1+g_x^2)}{\nu+g} - 2g_x u_x\right]\upsilon_{x_2}$$

for $g \in h_{+}^{2,\alpha}$ *and* $u,\, \upsilon \in Y$.

Proof. It is clear that $(\varphi,\, \upsilon) \to \varphi\upsilon$ is bilinear and continuous from h^s into Y. Moreover, the function

$$g \to \frac{1}{\nu+g}$$

is continuous and is infinitely differentiable from h^s into Y. By using the definition of the space Y and Lemma 2.3, we get that for all fixed $g \in h_{\Phi}^{2,\alpha}$ the operator $u \to B(g)$ is bounded linear operator from Y into X. So, we obtain that

$$B(.) \in C^{\infty}(h_{\Phi}^{2,\alpha},\, L(Y,\, X)).$$

Hence, in view of Lemma 2.3, we obtain

$$B_0(.) \in C^{\infty}(h_{\Phi}^{2,\alpha},\, L(Y,\, E)).$$ □

By using [4, Theorem 2] we obtain the following:

Theorem 3.1. *Suppose that Condition* 3.1 *is satisfied and* $\alpha \in (0, 1)$. *Then for* $\lambda \in S(\varphi)$ *and for sufficiently large* $|\lambda|$:

(a) *the operator* $u \to \tilde{O}(g)u = \{(B(g)+\lambda)u,\, \gamma_0 u,\, (\eta+g)B_1(g)\}$ *is isomorphism from* Y *onto* $X \times h^{2,\alpha}(A) \times h^{1,\alpha}(A)$;
(b) *for* $\mu > 0$ *the operator* $u \to \{(B(g)+\lambda)u,\, \mu\gamma_0 u,\, (\eta+g)B_1(g)\}$ *is isomorphism from* Y *onto* $X \times h^{2,\alpha}(A) \times h^{1,\alpha}(A)$;
(c) *for* $u \in Y$ *there exists a positive constant* C, *depending only on* g, η, p *and* E, *such that the coercive estimate holds*:

$$\|u\|_Y \leqslant C(\|(B(g)+\lambda)u\|_X + \|\gamma_0 u\|_{h^{2,\alpha}(E)} + \|(\nu+g)B_1 u\|_{h^{1,\alpha}(E)}). \quad (3.3)$$

Proof. Indeed, since the domain Q is a strip, g and η are fixed smooth functions. By virtue of trace theorem in Hölder space [16, §2], we obtain the assertion. □

Consider now, the following BVP

$$-\sum_{i,j=1}^{2} a_{ij}(g)\frac{\partial^2 u}{\partial x_i \partial x_j} + a_2(g)\frac{\partial u}{\partial x_2} + A(g)u = F, \tag{3.4}$$

$$B_i u = \sum_{j=1}^{2} b_{ji}(g)\gamma_i \frac{\partial u}{\partial x_j} = 0, \quad i = 0,\, 1, \tag{3.5}$$

$$-\sum_{i,j=1}^{2} a_{ij}(g)\frac{\partial^2 u}{\partial x_i \partial x_j} + a_2(g)\frac{\partial u}{\partial x_2} + A(g)u = 0, \tag{3.6}$$

$$\sum_{j=1}^{2} b_{j0}(g)\gamma_0 \frac{\partial u}{\partial x_j} = \psi, \quad \sum_{j=1}^{2} b_{j1}(g)\gamma_1 \frac{\partial u}{\partial x_j} = 0. \tag{3.7}$$

Consider the operators $\tilde{S}(g)$ and $\tilde{K}(g)$ generated by problems (3.4)–(3.5) and (3.6)–(3.7), respectively, i.e.,

$$D(\tilde{S}(g)) = \{u \in Y,\, B_i u = 0,\, i = 0,\, 1\},$$

$$\tilde{S}(g)u = -\sum_{i,j=1}^{2} a_{ij}(g)\frac{\partial^2 u}{\partial x_i \partial x_j} + a_2(g)\frac{\partial u}{\partial x_2} + A(g)u$$

and

$$D(\tilde{K}(g)) = \{u \in Y,\, Bu = 0,\, B_1 u = 0\}, \quad \tilde{K}(g)u = B_0 u \in B_{0p}.$$

From Theorem 3.1 we get that the inverse operators $\tilde{O}^{-1}(g)$, $\tilde{S}^{-1}(g)$, $\tilde{K}^{-1}(g)$ are bounded from $X \times h^{2,\alpha}(A) \times h^{1,\alpha}(A)$, X, $h^{2,\alpha}(E)$ into Y, respectively.

Put

$$O(g) = \tilde{O}^{-1}(g), \quad S(g) = \tilde{S}^{-1}(g), \quad K(g) = \tilde{K}^{-1}(g).$$

Assume $g \in h_+^{2,\alpha}(\mathbb{R};\, E)$, $\psi \in D_A(\alpha)$, and put $u = K(g)\upsilon$. Then u is the solution of the BVP (3.6)–(3.7).

Condition 3.2. Assume $A(g)$ is Gateaux differentiable for $g \in h_+^{2,\alpha}$, $\alpha \in (0,\, 1)$ and the operator $\partial A(g)$ is uniformly R-positive in UMD Banach algebra E.

Lemma 3.2. *Suppose that Conditions* 3.1 *and* 3.2 *are satisfied. Then, we have*

$$K(.) \in C^\infty(h_+^{2+\alpha};\, L(E,\, Y))$$

and

$$\partial K(g)[\upsilon,\, \psi] = -S(g)\partial B(g)[\upsilon,\, K(g)\psi]$$

for $g \in h_+^{2,\alpha}$, $\psi \in E$ *and* $\upsilon \in Y$.

Proof. It follows from Lemma 3.1 and Theorem 3.1 that the map $g \to \tilde{O}(g)$ is an isomorphism from Y onto $X \times h^{2,\alpha}(A) \times h^{1,\alpha}(A)$ and has continuous derivatives of any order with respect to $g \in h_+^{2,\alpha}$, i.e.,

$$\tilde{O} \in C^\infty(h_+^{2,\alpha};\, \mathrm{Isom}Y,\, X \times h^{2,\alpha}(A) \times h^{1,\alpha}(A))$$

and

$$\partial \tilde{O}(g)\psi = \{\partial B(g)[\psi,\, .],\, 0,\, 0\} \quad \text{for } \psi \in h^{2,\alpha}(A).$$

Then by reasoning as Lemma 3.4 in [9], we obtain the assertion. □

4. The Nonlinear Operator for Free BVPs

In this section we introduce the basic nonlinear operator and we derive some properties of it.

Moreover, we show that the corresponding evolution problem involving this operator is equivalent to the original problem (1.2). Given $g \in h_{+}^{2,\alpha}$, we define the following operator

$$O(g) = B_0(g)K(g)g.$$

From Lemmas 3.1 and 3.2, we get that

$$O \in C^{\infty}(h_{+}^{2,\alpha},\, h^{1,\alpha}(E)). \tag{4.1}$$

Assume that $g_0 \in h_{+}^{2,\alpha}$, and let $\sigma = [0,\, T)$. A function $\sigma \to B_{10}$ is a classical solution of

$$\frac{dg}{dt} + O(g) = 0, \quad g(0) = g_0 \tag{4.2}$$

iff $g \in C(\sigma;\, h_{+}^{2,\alpha}) \cap C^1(\sigma;\, h^{2,\alpha}(E))$ and g satisfies (4.2) pointwise.

Lemma 4.1. *Assume that Condition* 3.1 *is satisfied. Then for* $g_0 \in h_{+}^{2,\alpha}$:

(a) *if g is a classical solution of problem* (4.2) *on σ and if*

$$\upsilon(t,\,.) = K(g(t,\,.))g(t,\,.),$$

then the pair $(\upsilon,\, g)$ is a classical solution of (2.5) *on σ, having the additional regularity*

$$\begin{aligned} g \in C(\sigma;\, h_{U}^{2,\alpha}) \cap C^1(\sigma;\, h^{1,\alpha}), \\ \upsilon(t,\,.) \in Y, \quad t \in \sigma; \end{aligned} \tag{4.3}$$

(b) *if $(\upsilon,\, g)$ is a classical solution of* (2.5) *on σ having the regularity* (4.3), *then g is a classical solution of* (4.2) *on σ.*

Proof. The proof is obtained from Lemma 2.2 and definitions of the spaces B_{ip}, $i = 0,\, 1$ and Y. For fixed $g \in h_{\Phi}$ consider the operator

$$\upsilon \to B_0(g)K(g)\upsilon. \tag{4.4}$$

In view of Theorem 3.1, we have

$$B_0(g)K(g) \in L(h^{2,\alpha}(A),\, h^{1,\alpha}(A)). \tag{4.5}$$

□

Lemma 4.2. *If Conditions* 3.1 *and* 3.2 *are satisfied, then* $O \in C^{\infty}(h_{+}^{2,\alpha},\, h^{1,\alpha}(A))$ *and*

$$\partial O(g)\psi = B_0(g)K(g)\psi + \partial B_0(g)[\upsilon,\, K(g)\psi] - B_0(g)S(g)\partial B(g)[\upsilon,\, K(g)\psi] \tag{4.6}$$

for $g \in h_{+}^{2,\alpha}$, $\psi \in h^{2,\alpha}(A)$ *and* $\upsilon \in Y$.

Proof. The assertion is obtained from Lemma 3.1 and Lemma 3.2. □

5. Linear Equation with Constant Coefficients

We put
$$\mathbb{R}^2_+ = \{x = (x_1,\, x_2) \in \mathbb{R}^2,\, x_2 > 0\}.$$
Consider the problem
$$-\sum_{i,j=1}^{2} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + Au + \mu^2 u = 0, \tag{5.1}$$
$$u(x_1,\, 0) = \psi(x_1), \quad x \in \mathbb{R}^2_+, \tag{5.2}$$
where $A = A(g)(x_0,\, 0)$, $a_{ij} = a_{ij}(g)(x_0,\, 0)$ and $a_{ij}(g)$ are defined by (2.7) and $\psi \in h^{2,\alpha}(A)$. By applying the Fourier transform to the problem (5.1)–(5.2) with respect to x_1, we get
$$-a_{22}\frac{d^2\hat{u}}{dy^2} + 2a_{12}i\eta\frac{d\hat{u}}{dy} + (A + \eta^2 + \mu^2)\hat{u} = 0, \tag{5.3}$$
$$\hat{u}(\eta,\, 0) = \hat{\psi}(\eta), \quad \eta \in \mathbb{R},\, y \in \mathbb{R}_+, \tag{5.4}$$
where x_2 and $\hat{u}(\eta,\, x_2)$ are denoted by y and $\hat{u} = \hat{u}(\eta,\, y)$, respectively.

Let
$$p(\xi) = \xi_1^2 + 2a_{12}\xi_1\xi_2 + a_{22}\xi_2^2 \quad \text{for } \xi = (\xi_1,\, \xi_2) \in \mathbb{R}^2.$$
There exists an $\alpha > 0$ with
$$p(\xi) \geqslant \alpha|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2. \tag{5.5}$$
Condition (5.5) implies that
$$a_{12}^2 - a_{22} \geqslant \alpha. \tag{5.6}$$
Moreover, we define
$$q_\eta(\mu,\, \lambda) = -a_{22}\lambda^2 + 2a_{12}i\eta\lambda + \mu^2 + \eta^2 + 1 = 0. \tag{5.7}$$

Remark 5.1. Given $\eta \in \mathbb{R}$ and $z \in \mathbb{C}$, then in view of (5.5) there is exactly one root of the equation (5.7) with positive real part. It is given by
$$\lambda(\eta,\, \mu) = ia(\eta) + b(\eta,\, \mu), \tag{5.8}$$
where
$$a(\eta) = -\frac{a_{12}}{a_{22}}\eta,\; b(\eta,\, \mu) = \frac{1}{a_{22}}[a_{22}(1+\mu^2) + (a_{12}^2 - a_{22})\eta^2]^{\frac{1}{2}}.$$

We put
$$\begin{gathered}\tilde{X} = h^\alpha(\mathbb{R}^2_+;\, E), \quad \tilde{Y} = h^{2,\alpha}(\mathbb{R}^2_+;\, E(A);\, E), \\ N_\mu(\eta,\, y) = \exp\{-\lambda(\eta,\, \mu)A_\mu^{\frac{1}{2}}(\eta)y\}.\end{gathered} \tag{5.8.0}$$

The main result of this section is the following:

Theorem 5.2. *Assume that Condition* 3.1 *is satisfied. Then problem* (5.1)–(5.2) *has a unique solution* $u \in \tilde{Y}$ *for* $\psi \in h^{2,\alpha}(A)$ *and* u *is represented by*
$$u(x_1,\, x_2) = (\tilde{K}_0 + \mu^2)\psi = F^{-1}N_\mu(\eta,\, x_2)F\psi(\eta).$$

Moreover, the following estimate holds:

$$\sum_{j=0}^{2}|\mu|^{2-j}\left\|\frac{\partial^j u}{\partial x_1^j}\right\|_{\tilde{X}}+\sum_{i,j=1}^{2}\left\|\frac{\partial^2 u}{\partial x_i\partial x_j}\right\|_{\tilde{X}}+\|Au\|_{\tilde{X}}\leqslant C\|\psi\|_{h^{2,\alpha}(A)}. \tag{5.9}$$

For the proof we need some preparation. Put

$$A_\mu=A_\mu(\eta)=A+\mu^2+\eta^2,\quad q_0(\eta,\,y)=AN_\mu(\eta,\,y),$$
$$q_j(\eta,\,y)=|\mu|^{2-j}\eta^j N_\mu(\eta,\,y).$$

We need the following lemmas.

Lemma 5.1. *Assume that Condition* 3.1 *is satisfied. Then there exists a unique solution* $\hat{u}(\eta,\,y)$ *of* (5.3)–(5.4) *expressing as*

$$\hat{u}(\eta,\,y)=N_\mu(\eta,\,y)\hat{\psi}(\eta). \tag{5.10}$$

Moreover, the following estimate holds uniformly in η:

$$\|A\hat{u}\|_{h^\alpha(\mathbb{R}_+;E)}+\sum_{j=0}^{2}|\mu|^{2-j}\|\hat{u}^{(j)}\|_{h^\alpha(\mathbb{R}_+;E)}\leqslant C\|\hat{\psi}\|_{h^{2,\alpha}(A)}. \tag{5.11}$$

Proof. In view of the positivity property of operator A we know that $-A_\mu^{\frac{1}{2}}(\eta)$ is an analytic semigroup in E (see for example [24, §1.18]). Then the equation (5.3) has a solution $\hat{u}=U_{\eta,\mu}(y)\hat{\psi}(\eta)$ on $(0,\,\infty)$, where

$$U_{\eta,\mu}(y)=\exp\{-\lambda(\eta,\,\mu)A_\mu^{\frac{1}{2}}(\eta)y\}\hat{\psi}(\eta)$$

and $\lambda(\eta,\,\mu)$ is a root of (5.7) with positive real part. From the above expression and the properties of analytic semigroups, we get the uniform estimate

$$\begin{aligned}\|A\hat{u}\|_{h^\alpha(\mathbb{R}_+;E)}&+\sum_{j=0}^{2}|\mu|^{2-j}\|\hat{u}^{(j)}\|_{h^\alpha(\mathbb{R}_+;E)}\\&\leqslant C_1\sup_{\eta\in[0,1]}\|\eta^{1-\alpha}AU(\eta)\hat{\psi}(.,\,\eta)\|_{h^\alpha(\mathbb{R}_+;E)}\leqslant C\|\hat{\psi}\|_{h^{2,\alpha}(A)},\end{aligned}$$

where $U(y)$ is a semigroup generated by $-A$. □

Lemma 5.2. *Assume that Condition* 3.1 *is satisfied. Then operator functions* $q_0(\eta,\,y)$ *and* $q_j(\eta,\,y)$ *are Fourier multipliers in* $h^\alpha(\mathbb{R};\,E)$ *uniformly with respect to* $y\in\mathbb{R}_+$.

Proof. In view of (5.6) and Remark 5.1, we get

$$\sup_{\eta\in\mathbb{R},y\in\mathbb{R}_+}(1+|\eta|)^{\frac{1}{2}}\left\|\frac{d}{d\eta}[q_0(\eta,\,y)]\right\|_{B(E)}\leqslant C_1,$$
$$\sup_{\eta\in\mathbb{R},y\in\mathbb{R}_+}(1+|\eta|)^{\frac{1}{2}}\left\|\frac{d}{d\eta}[q_j(\eta,\,y)]\right\|_{B(E)}\leqslant C_2.$$

Then in a similar way as in [27, Ch. 4, p. 96] we can prove that the operator function $q_0(.,\,y)$, $q_j(.,\,y)$ are multipliers in $h^\alpha(\mathbb{R};\,E)$ uniformly in $y\in\mathbb{R}_+$ and $\mu\in\mathbb{R}$. □

Let

$$\Phi_0(y) = \|q_0(.,\, y)\|_{M(h^\alpha(\mathbb{R};E))}, \qquad \Phi_j(y) = \|q_j(.,\, y)\|_{M(h^\alpha(\mathbb{R};E))}.$$

Lemma 5.3. *Assume that Condition* 3.1 *is satisfied. Then*

$$\Phi_0(y) \to 0,\ \Phi_j(y) \to 0 \ \text{as}\ y \to \infty.$$

Proof. Indeed, by properties of Fourier multiplier operators from in $h^\alpha(\mathbb{R};\, E)$ and the theory of analytic semigroups there exists $\omega > 0$ such that

$$\begin{aligned} \|q_0(.,\, y)\|_{M(h^\alpha(\mathbb{R};E))} &\leqslant C_1(|\lambda(\eta,\, \mu)|y)^{-1} \exp\{-\omega|\lambda(\eta,\, \mu)|y\}, \\ \|q_j(.,\, y)\|_{M(h^\alpha(\mathbb{R};E))} &\leqslant C_2|\mu|^{2-j}(1+\eta^2)^{\frac{1}{2}} \exp\{-\omega|\lambda(\eta,\, \mu)|y\}. \end{aligned} \tag{5.12}$$

By estimates (5.12) and Remark 5.1, we obtain the assertion. □

Proof of Theorem 5.1. By Lemma 5.1 the problem (5.1)–(5.2) has a solution

$$u(x_1,\, x_2) = (\tilde{K}_0 + \mu^2)\psi = F^{-1} N_\mu(\eta,\, x_2)\hat{\psi}(\eta).$$

By Lemmas 5.2, 5.3 the operator-functions $q_0(\eta,\, y)$ and $q_j(\eta,\, y)$ are Fourier multipliears in $h^\alpha(\mathbb{R};\, E)$ uniformly with respect to $y \in \mathbb{R}_+$. Then from the estimate (5.12), we have the assertion. □

Let $K_{0\mu} = K_{0\mu}(x_0)$ denote the inverse of the operator $\tilde{K}_0(x_0) + \mu^2$, i.e.,

$$K_{0\mu}(x_0) = [\tilde{K}_0(x_0) + \mu^2]^{-1}.$$

Result 5.1. From Theorem 5.1 we deduced that the opertor $u \to K_{0\mu}u$ is bounded from $h^{2,\alpha}(A)$ into $\tilde{Y}$ and the following estimate holds:

$$\sum_{j=0}^{2} |\mu|^{2-j} \left\| \frac{\partial^j K_{0\mu}}{\partial x_1^j} \right\|_{B(h^{2,\alpha}(A),\tilde{X})} + \sum_{i,j=1}^{2} \left\| \frac{\partial^2 K_{0\mu}}{\partial x_i \partial x_j} \right\|_{B(h^{2,\alpha}(A),\tilde{X})} + \|AK_{0\mu}\|_{B(h^{2,\alpha}(A),\tilde{X})} \leqslant C.$$

Consider now the BVP

$$\begin{aligned} B(x_0)u &= \sum_{i,j=1}^{2} -a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + Au + \mu^2 u = 0, \qquad x \in \mathbb{R}^2_+, \\ B_0(x_0)u &= \left[b_1 \frac{\partial}{\partial x_1} u(x) + b_2 \frac{\partial}{\partial x_2} u(x) \right] \Big|_{x_2=0} = \psi(x_1), \end{aligned}$$

where

$$A = A(g)(x_0,\, 0), \qquad a_{ij} = a_{ij}(g)(x_0,\, 0), \qquad b_i = b_{ij}(g)(x_0,\, 0) \tag{5.13}$$

and $a_{ij}(g)$ are defined by (2.7).

6. Regularity Properties of Abstract Elliptic Operator with Constant Coefficients

Consider the operator

$$u \to O_1 u = u \to O_1(g)u = B_0(g)K(g)u.$$

Here, O_{10} denotes the constant coefficients version of O_1 fixed in $(x_0,\, 0)$ by (5.13), i.e.,

$$u \to O_{10}u = O_1(x_0,\, 0) = B_0 K(g)(x_0,\, 0)u.$$

Here,

$$O_{1\mu}u = (O_1 + \mu^2)u, \quad O_{10\mu}u = (O_{10} + \mu^2)u,$$
$$K_0 = K(g(x_0,\, 0)), \quad K_\mu(g) = [\tilde{K}(g) + \mu^2]^{-1}.$$

Put

$$a_1(\eta,\, \mu) = ib_1(x_0)\eta - b_2(x_0)\lambda(\eta,\, \mu).$$

We show the following result:

Theorem 6.1. *Assume that Condition* 3.1 *is satisfied. Then the operator* $u \to (O_{10} + \mu^2)u$ *is an isomorphism from* $h^{2,\alpha}(A)$ *onto* $h^{1,\alpha}(A)$*. Moreover, the uniform estimates*

$$\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_1^i} \right\|_{h^{1,\alpha}(A)} + \|Au\|_{h^{1,\alpha}(A)} \leqslant C\|(O_{10} + \mu^2)u\|_{h^{2,\alpha}(A)},$$
$$C_1\|u\|_{h^{2,\alpha}(A)} \leqslant \|O_{20} + \mu_0^2 u\|_{h^{1,\alpha}(A)} \leqslant C_2\|u\|_{h^{2,\alpha}(A)} \tag{6.1}$$

hold for all $u \in h^{2,\alpha}(A)$ *and* $\mu > 0$ *with sufficiently large* μ_0.

Proof. In view of Lemma 3.1 and Theorem 5.1 it is clear to see that the problem

$$(O_{10} + \mu^2)u = \psi$$

has a unique solution u exspressed as

$$u = F^{-1}a_1(\eta,\, \mu)N_\mu(\eta,\, x_2)F\psi(\eta).$$

By Lemmas 5.1–5.3 and by multiplier result in [5], we get that the operator functions $|\mu|^{2-i}\eta^i K(\eta,\, x_2)$ and $AK(\eta,\, x_2)$ are multipliers in $h^\alpha(\mathbb{R};\, E)$ uniformly with respect to $x_2 \in \mathbb{R}_+$, where

$$K(\eta,\, x_2) = a_1(\eta,\, \mu)N_\mu(\eta,\, x_2).$$

Hence, we obtain the estimate

$$\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_1^i} \right\|_{h^{1,\alpha}(A)} + \|A_0 u\|_{h^{1,\alpha}(A)} \leqslant C\|(O_{10} + \mu^2)u\|_{h^{2,\alpha}(A)}.$$

Moreover, by definitions of the space $h^{2,\alpha}(A)$ we get (6.1) and the estimate

$$\|O_{10} + \mu_0^2 u\|_{h^{1,\alpha}(A)} \leqslant C_2\|u\|_{h^{2,\alpha}(A)}.$$

Then in view of the above estimate, by reasoning as in Theorem 5.1 we get the assertion and corresponding esimates. □

By reasoning as in Theorem 6.1 we obtain

Theorem 6.2. *Assume that Condition* 3.1 *is satisfied. Then the operator* O_{10} *is positive and* $-O_{10}$ *is a generator of an analytic semigroup in* $h^{1,\alpha}(A)$.

Proof. Indeed, for positivity of the operator O_{10} in $h^{1,\alpha}(A)$ we need to show the estimate

$$\|(O_{10}+\mu^2)^{-1}\|_{B(h^{1,\alpha}(A))} \leqslant C\mu^{-2},$$

i.e., we have to prove the estimate

$$\|u\|_{h^{2,\alpha}(A)} \quad \leqslant C\mu\|(O_{10}+\mu^2)^{-1}u\|_{h^{1,\alpha}(A)}$$

for $u \in h^{2,\alpha}(A)$. By reasoning as above, we get that the function $\mu^2\eta a_2(\eta,\, \mu)$ is a multiplier in $h^{\alpha}(\mathbb{R};\, E)$ uniformly with respect to $x_2 \in \mathbb{R}_+$, where

$$a_2(\eta,\, x_2) = a_1(\eta,\, \mu)N_{\mu}(\eta,\, x_2).$$

So, the operator O_{10} is positive in $h^{1,\alpha}(A)$. Then by [24, §1.18] we obtain that $-O_{10}$ is a generator of an analytic semigroup in $h^{1,\alpha}(A)$. □

Here

$$Q_2u = O_2(g)u = \partial B_0(g)[u,\, K(g)g] \tag{6.2}$$

for $g \in h_+^{2+\alpha}$ and $u \in h^{2,\alpha}(A)$.

Consider the operator O_{20} that is a constant coefficients version of O_2 with fixed point $(x_0,\, 0)$ by (5.13), i.e.,

$$O_{20} = O_{20}(g) = \partial B_0(g)[.,\, K(g)g](x_0,\, 0) \tag{6.3}$$

for $g \in h^{2,\alpha}(A)$.

We prove the following result:

Theorem 6.3. *Assume that Conditions* 3.1 *and* 3.2 *are satisfied. Then the operator* $u \to (O_{20}+\mu^2)u$ *is an isomorphism from* $h^{2,\alpha}(A)$ *onto* $h^{1,\alpha}(A)$. *Moreover, the following estimates hold*:

$$\begin{gathered}\sum_{i=0}^{1}|\mu|^{2-i}\left\|\frac{\partial^i u}{\partial x_1^i}\right\|_{h^{1,\alpha}(A)} + \|A_0u\|_{h^{1,\alpha}(A)} \leqslant C\|(O_{20}+\mu^2)u\|_{h^{2,\alpha}(A)},\\ C_1\|u\|_{h^{2,\alpha}(A)} \leqslant \|O_{20}+\mu_0^2u\|_{h^{1,\alpha}(A)} \leqslant C_2\|u\|_{h^{2,\alpha}(A)}\end{gathered} \tag{6.4}$$

for all $u \in h^{2,\alpha}(A)$, *for sufficiently large* μ_0 *and* $\mu > 0$.

Proof. By Lemma 3.1, we have

$$\begin{gathered}O_{20}u = -\frac{\partial \upsilon}{\partial x_1}u_{x_1} + \frac{\partial \upsilon}{\partial x_2}(\Lambda_1u+\Lambda_2u), \quad \text{where}\\ \upsilon = \upsilon(g) = K(g)g, \quad \Lambda_1u = \frac{(1+g_{x_1}^2)u}{\nu+g}, \quad \Lambda_2u = -\frac{1}{\nu+g}2g_{x_1}u_{x_1},\\ \upsilon_0 = \upsilon_0(g) = K(g)(x_0,\, 0)g,\end{gathered} \tag{6.5}$$

By using Lemmas 5.1–5.3 we get that the operator function

$$(\eta^2 + \mu^2 + A_{0\mu})[\eta + \Lambda(u)a_1(\eta, \mu)]N_{0\mu}(\eta, x_2)$$

is a multiplier in $h^\alpha(\mathbb{R}; E)$ uniformly with respect to $x_2 \in \mathbb{R}_+$, where

$$A_{0\mu} = A_{0\mu}(\eta) = A + \mu^2 + \eta^2,\ N_{0\mu}(\eta, y) = A_{0\mu}^{-\frac{1}{2}}(\eta)\exp\{-\lambda(\eta, \mu)A_{0\mu}^{\frac{1}{2}}(\eta)y\}. \quad (6.6)$$

Then by reasoning as in Theorem 6.1 we obtain the assertion. □

From Theorem 6.3, we obtain

Result 6.1. Assume that Conditions 3.1 and 3.2 are satisfied. Then the operator O_{20} is positive and $-O_{20}$ is a generator of an analityc semigroup in $h^{1,\alpha}(A)$.

Consider first of all, the BVP

$$-\sum_{k,j=1}^{2} a_{kj}\frac{\partial^2 u}{\partial x_k \partial x_j} + (A + \mu^2)u + \mu^2 u = V(x), \quad (6.7)$$

$$u(x_1, 0) = 0, \quad x = (x_1, x_2) \in \mathbb{R}_+^2, \quad (6.8)$$

where $A = A(g)(x_0, 0)$, $a_{kj} = a_{kj}(g)(x_0, 0)$ and $a_{kj}(g)$ are defined by (5.13).

Let $S_0 = S(g)(x_0, 0)$ denotes the realizasion operator in $\tilde{X}$ generated by (6.7)–(6.8) for $\mu = 0$, i.e.,

$$D(S_0) = \tilde{Y},\ S_0 u = -\sum_{k,j=1}^{2} a_{kj}\frac{\partial^2 u}{\partial x_k \partial x_j} + Au.$$

From [4, Theorem 2], we obtain the following:

Result 6.2. Assume that Condition 3.1 is satisfied. Then;

(1) problem (6.7)–(6.8) for sufficiently large $\mu > 0$ has a unique solution $u \in \tilde{Y}$ for $V \in \tilde{X}$;

(2) the uniform coercive estimate holds:

$$\sum_{i=0}^{1} |\mu|^{2-i}\left\|\frac{\partial^i u}{\partial x_1^i}\right\|_{\tilde{X}} + \sum_{k,j=1}^{2}\left\|\frac{\partial^2 u}{\partial x_k \partial x_j}\right\|_{\tilde{X}} + \|Au\|_X \leqslant C\|V\|_{\tilde{X}}; \quad (6.9)$$

(3) the operator S_0 is a positive and $-S_0$ is a generator of an analytic semigroup in $h^{1,\alpha}(A)$.

The estimate (6.9) particularly, implies that $(S_0 + \mu^2)^{-1} \in L(\tilde{X}, \tilde{Y})$.

Consider the inhomogeneous problem

$$-\sum_{k,j=1}^{2} a_{ij}\frac{\partial^2 u}{\partial x_k \partial x_j} + Au + \mu^2 u = V(x), \quad (6.10)$$

$$\gamma u = u(x_1, 0) = \psi(x_1), \quad x = (x_1, x_2) \in \mathbb{R}_+^2. \quad (6.11)$$

Theorem 6.4. *Assume that Conditions* 3.1 *and* 3.2 *are satisfied. Then the operator* $u \to G_0 u = \{L_0 u, \gamma u\}$ *is an isomorphism from* $\tilde{Y}$ *onto* $\tilde{X} \times h^{2,\alpha}(A)$.

Proof. From definition of $\tilde{X}$, $\tilde{Y}$, $h^{2,\alpha}(A)$, from expresion of L_0 and by virtue of trace result in $\tilde{Y}$ [16, Ch. 2], we get

$$\|G_0 u\|_{\tilde{X}\times h^{1,\alpha}(A)} = \|L_0 u\|_{\tilde{X}} + \|\gamma u\|_{h^{2,\alpha}(A)} \leqslant C\|u\|_{\tilde{Y}},$$

i.e., the operator G_0 is bounded linear from $\tilde{Y}$ into $\tilde{X}\times h^{2,\alpha}(A)$. Hence, in view of Banach theorem it is sufficient to show that the operator G_0 is injective and surjective from $\tilde{Y}$ onto $\tilde{X}\times h^{2,\alpha}(A)$. From Theorem 5.1 we obtain that the corresponding homogenous problem $L_0 u = 0$, $\gamma u = 0$ has a zero solution, i.e., the operator G_0 is injective. So, it remain to show that this operator is surjective. By Theorem 5.1 we obtain that problem $L_0 u = 0$, $\gamma u = \psi$ has a solution $u_1 \in \tilde{Y}$ for all $\psi \in h^{2,\alpha}(A)$. Moreover, from Result 6.2 we get that the problem $L_0 u = V$, $\gamma u = 0$ has a solution $u_2 \in \tilde{Y}$ for all $V \in \tilde{X}$. Then $u = u_1 + u_2$ is a solution of (6.10)–(6.11) that belongs to $\tilde{Y}$, i.e., the operator G_0 is surjective from $\tilde{Y}$ onto $\tilde{X}\times h^{2,\alpha}(A)$. □

From Theorems 5.1 and 6.4 we obtain the following

Result 6.3. The solution u of the problem (6.10)–(6.11) is expressed as

$$u(x) = S_{1,\mu}V + S_{2,\mu}(\psi - \gamma u_1),$$

where

$$S_{1,\mu}V = r_+ F^{-1}(p(\xi) + A + \mu^2)^{-1} F\tilde{V}, \quad S_{2,\mu}\upsilon = F^{-1}N_\mu(\xi_1, x_2)F\upsilon,$$

here r_+ is the restriction operator from $\mathbb{R}^2$ into $\mathbb{R}^2_+$ and $\tilde{V} = \tilde{V}(x_1, x_2)$ is an exstension of $V(x_1, x_2)$ on $\mathbb{R}^2$, i.e.

$$\tilde{V}(x_1, x_2) = \begin{cases} V(x_1, x_2), & \text{if } x_1, x_2 \in \bar{\mathbb{R}}^2_+, \\ V(x_1, -x_2), & \text{if } x_1, x_2 \in \mathbb{R}^2_+, \end{cases}$$

$N_\mu(\xi_1, x_2)$ is the operator function defined by (5.8.0) and u_1 is a solution of the equation

$$-\sum_{k,j=1}^{2} a_{ij}\frac{\partial^2 u}{\partial x_k \partial x_j} + Au + \mu^2 u = \tilde{V}(x), \quad x \in \mathbb{R}^2.$$

Let $O_3 = O_3(g)$ be the operator in (4.6) defined by

$$O_3 u = B_0(g)S(g)\partial B(g)[u, K(g)g] \tag{6.12}$$

for $g \in h^{2,\alpha}(A)$ and $u \in Y$. In view of Lemma 3.1 and Lemma 3.2 we get

$$O_3 u = B_0(g)S(g)\partial B(g)[u, \upsilon], \quad \upsilon = \upsilon_g(x) = (K(g)g)(x), \quad \text{where}$$

$$\begin{aligned} \partial B(g)[u, \upsilon] = \frac{2\beta}{\nu + g}\Big\{\Big[\Big(\frac{g_{x_1}u}{\nu+g} - u_{x_1}\Big)\Big]\upsilon_{x_1x_2} \\ + \frac{1}{(\nu+g)^2}\Big[\Big(\frac{1}{\beta} + \beta g_{x_1}^2\Big)u - \frac{\beta}{\nu+g}g_{x_1}u_{x_1}\Big]\upsilon_{x_2x_2} + \partial A(g)[u, \upsilon] \\ - \Big[\frac{g_{x_1}^2 u}{(\nu+g)^2} - \frac{g_{x_1x_1}u + 4g_{x_1}u_{x_1}}{2(\nu+g)} + \frac{u_{x_1x_1}}{2}\Big]\upsilon_{x_2}\Big\}, \end{aligned} \tag{6.13}$$

and

$$O_3 = O_3(g) = \sum_{i=1}^{4} O_{3i}(g)u, \quad O_{3i}(g)u = B_0(g)S(g)G_i(g), \quad \text{where}$$

$$G_i(g) = G_i(g)[u, \upsilon], \quad G_1(g) = \frac{2\beta}{\nu+g}\Big[\Big(\frac{g_{x_1}u}{\nu+g} - u_{x_1}\Big)\Big]\upsilon_{x_1x_2},$$

$$G_2(g) = \Big[\frac{u}{(\nu+g)^2}\Big(\frac{1}{\beta} + \beta g_x^2\Big) - \frac{\beta}{\nu+g}g_{x_1}u_{x_1}\Big]\upsilon_{x_2x_2}, \tag{6.14}$$

$$G_3(g) = \frac{2\beta}{\nu+g}\partial A(g)[u, \upsilon],$$

$$G_4(g) = -\Big[\Big(\frac{g_{x_1}^2}{(\nu+g)^2} - \frac{g_{x_1x_1}}{2(\nu+g)}\Big)u - \frac{4g_{x_1}u_{x_1}}{2(\nu+g)} + \frac{u_{x_1x_1}}{2}\Big]\upsilon_{x_2}.$$

Consider the operator O_{30} that is the constant coefficients version of O_3 fixed in $(x_0, 0)$ which is defined by (6.12), i.e., from the above equality and from (6.13) we get $O_{3ik} = B_0(g)S(g)G_k(g)(x_0, 0)$,

$$O_{30}u = O_{30}(g)u = O_{301}u + O_{302}u + O_{303}u + O_{304}u, \tag{6.15}$$

where

$$O_{30i} = O_{3i}(g(x_0, 0))u.$$

Put

$$w(x) = w_g(x) = \frac{\beta}{\nu+g}\frac{\partial}{\partial x_2}\upsilon_g(x_1, x_2), \quad (x_1, x_2) \in \Omega, \quad w_0 = w_g(x^0, 0).$$

For $u \in h^{2,\alpha}(A)$ define the operator by

$$P_1u(x_1, x_2) = (P_1(g, x_1^0)u)(x_1, x_2) = w_g(x)u_{x_1x_1}e^{-x_2}.$$

For later purposes we need the following technical lemmas.

Lemma 6.1. *Assume that Conditions* 3.1 *and* 3.2 *are satisfied. Then*:

(a) $P_1 \in B(h^{2,\alpha}(A), \tilde{X})$;

(b) *There exists a positive constant $C = C(g)$ such that*

$$\|P_1u - wu_{x_1x_1}\|_{h^\alpha(\tilde{U}_r;E)} \leqslant Cr\|u\|_{h^{2,\alpha}(A)}$$

for all $u \in h^{2,\alpha}(A)$, $r \in (0, 1)$, where

$$\tilde{U}_r = \mathbb{R}_+^2 \cap U_r(x_1^0, 0)$$

and $U_r(x_1^0, 0)$ denotes the two-dimensional ball with radius r centered at $(x_1^0, 0)$.

Proof. Indeed, from the identities (6.14) in view of Theorem 5.1 and by virtue of the trace theorem in Y we get (a); then by using the integral mean value theorem and the trace theorem we obtain

$$\|P_1u - wu_{x_1x_1}\|_{h^\alpha(\tilde{U}_r;E)} \leqslant \|(we^{-x_2} - w)u_{x_1x_1}\|_{h^\alpha(\tilde{U}_r;E)} \leqslant Cr\|u\|_{h^{2,\alpha}(A)}. \qquad \square$$

Let

$$\partial A = \partial A(x) = \partial A(g)(x), \; A_0 = \partial A(g)(x_{10}, 0).$$

Lemma 6.2. *Assume that Conditions* 3.1 *and* 3.2 *are satisfied. Then the operator* $u \to (O_{304} + \mu^2)u$ *is an isomorphism from* $h^{2,\alpha}(A)$ *onto* $h^{1,\alpha}(A)$*. Moreover, the following estimates hold*:

$$\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_1^i} \right\|_{h^{1,\alpha}(A)} + \|A_0 u\|_{h^{1,\alpha}(A)} \leqslant C\|(O_{304} + \mu^2)u\|_{h^{2,\alpha}(A)},$$
$$C_1\|u\|_{h^{2,\alpha}(A)} \leqslant \|O_{304} + \mu_0^2 u\|_{h^{1,\alpha}(A)} \leqslant C_2\|u\|_{h^{2,\alpha}(A)} \tag{6.16}$$

for all $u \in h^{2,\alpha}(A)$ *and for sufficiently large* μ_0, $\mu > 0$.

Proof. From the identities (6.14) and Lemma 6.1 by reasoning as in [9, Lemma 5.3] we get that the operator $u \to (P_1 + \mu^2)u$ is an isomorphism from $\tilde{Y}$ onto $\tilde{X}$. The Theorem 6.4 implies that the operator $u \to [S_{x_0}(g) + \mu^2]G_{40}u$ is an isomorphism from $\tilde{Y}$ onto $\tilde{Y}$ for sufficiently large $\mu > 0$. Then in view of the trace theorem in $\tilde{Y}$ we get that the operator $u \to [O_{304} + \mu^2]u$ is an isomorphism from $\tilde{Y}$ onto $h^{1,\alpha}(A)$. Moreover, in view of Result 6.3 by reasoning as in Theorem 6.1 we obtain the estimates (6.16) for all $u \in h^{2,\alpha}(A)$ and for $\mu > 0$ with sufficiently large μ_0. □

Lemma 6.3. *Assume that Conditions* 3.1 *and* 3.2 *are satisfied. Then the operator* $u \to (O_{303} + \mu^2)u$ *is an isomorphism from* $h^{2,\alpha}(A)$ *onto* $h^{1,\alpha}(A)$*. Moreover, the following estimates hold*:

$$\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_1^i} \right\|_{h^{1,\alpha}(A)} + \|A_0 u\|_{h^{1,\alpha}(A)} \leqslant C\|(O_{303} + \mu^2)u\|_{h^{2,\alpha}(A)},$$
$$C_1\|u\|_{h^{2,\alpha}(A)} \leqslant \|O_{303} + \mu_0^2 u\|_{h^{1,\alpha}(A)} \leqslant C_2\|u\|_{h^{2,\alpha}(A)} \tag{6.17}$$

for all $u \in h^{2,\alpha}(A)$ *and for sufficiently large* μ_0, $\mu > 0$.

Proof. From (6.14) by properties of positive operators we get that the map $u \to (G_{30} + \mu^2)u$ is an isomorphism from $\tilde{Y}$ onto $\tilde{X}$. The Theorem 6.4 implies that the operator $u \to [S_{x_0}(g) + \mu^2]G_{30}u$ is an isomorphism from $\tilde{Y}$ onto $\tilde{Y}$ for sufficiently large $\mu > 0$. Then in view of the trace theorem in $\tilde{Y}$ we get that the operator $u \to [O_{303} + \mu^2]u$ is an isomorphism from $\tilde{Y}$ onto $h^{1,\alpha}(A)$. Moreover, in view of Result 6.3 by reasoning as in Theorem 6.1 we obtain the estimates (6.17) for all $u \in h^{2,\alpha}(A)$ and for $\mu > 0$ with sufficiently large μ_0. □

In a similar way as Lemma 6.1 and by reasoning as in [9, lemma 5.4], we obtain

Lemma 6.4. *Assume that Conditions* 3.1 *and* 3.2 *are satisfied. Then the operator* $u \to (O_{30k} + \mu^2)u$ *is an isomorphism from* $h^{2,\alpha}(A)$ *onto* $h^{1,\alpha}(A)$*. Moreover, the following estimates hold*:

$$\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_1^i} \right\|_{h^{1,\alpha}(A)} + \|A_0 u\|_{h^{1,\alpha}(A)} \leqslant C\|(O_{30k} + \mu^2)u\|_{h^{2,\alpha}(A)},$$
$$C_1\|u\|_{h^{2,\alpha}(A)} \leqslant \|O_{30k} + \mu_0^2 u\|_{h^{1,\alpha}(A)} \leqslant C_2\|u\|_{h^{2,\alpha}(A)}, \quad k = 1, 2, \tag{6.18}$$

for all $u \in h^{2,\alpha}(A)$ *and* $\mu > 0$.

Theorem 6.5. *Assume that Conditions* 3.1 *and* 3.2 *are satisfied. Then the operator* $u \to (Q_{30} + \mu^2)u$ *is an isomorphism from* $h^{2,\alpha}(A)$ *onto* $h^{1,\alpha}(A)$*. Moreover, the following estimates hold*:

$$\begin{gathered} \sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_1^i} \right\|_{h^{1,\alpha}(A)} + \|A_0 u\|_{h^{1,\alpha}(A)} \leqslant C \|(O_{30} + \mu^2)u\|_{h^{2,\alpha}(A)}, \\ C_1 \|u\|_{h^{2,\alpha}(A)} \leqslant \|O_{30} + \mu_0^2 u\|_{h^{1,\alpha}(A)} \leqslant C_2 \|u\|_{h^{2,\alpha}(A)} \end{gathered} \tag{6.19}$$

for all $u \in h^{2,\alpha}(A)$ *and* $\mu > 0$.

Proof. From the identities (6.13)–(6.14) and from Lemmas 6.2–6.4 we get that the operator $u \to K_{1x_0}(g)u$ is an isomorphism from $\tilde{Y}$ onto $\tilde{X}$. Theorem 6.4 implies that the operator $u \to [S_{x_0}(g) + \mu^2]K_{1x_0}(g)u$ is an isomorphism from $\tilde{Y}$ onto $\tilde{Y}$ for sufficiently large $\mu > 0$. Then in view of trace theorem in $\tilde{Y}$ we get that the operator $u \to [O_{30} + \mu^2]u$ is an isomorphism from $\tilde{Y}$ onto $h^{1,\alpha}(A)$ for sufficiently large μ_0 and $\mu > 0$. □

From Theorem 6.5 we obtain

Result 6.4. Assume that Conditions 3.1 and 3.2 are satisfied. Then the operator O_{30} is positive and $-O_{30}$ is a generator of an analytic semigroup in $h^{1,\alpha}(A)$.

From Theorems 6.1, 6.3 and 6.5 we obtain

Result 6.5. Assume that Conditions 3.1 and 3.2 are satisfied. Then

$$O_{k0} + \mu_0 \in H(h^{2,\alpha}(A),\, h^{1,\alpha}(A)), \quad k = 1,\, 2,\, 3,$$

for sufficiently large $\mu_0 > 0$.

In view of Theorems 6.3, 6.5 and by Lemmas 2.3, 3.1 and Theorem 3.1 we obtain

Result 6.6. Assume that Conditions 3.1 and 3.2 are satisfied. The estimate

$$\|O_{20}(g) + O_{30}(g) + \mu_0\|_{B(h^{2,\alpha}(A),h^{1,\alpha}(A))} \leqslant C\|g\|_{h^{2,\alpha}(A)}$$

holds for all

$$g \in B_0 \subset h^{2,\alpha}(A).$$

Let

$$O_0 = O_{01} + O_{02} + O_{03}.$$

Now, we will show that

$$O_0 \in H(h^{2,\alpha}(A),\, h^{1,\alpha}(A)).$$

Theorem 6.6. *Assume that Conditions* 3.1 *and* 3.2 *are satisfied. Suppose*

$$\|w_0\|_{D_A(\alpha)} \leqslant \frac{\alpha(g)(x)}{a_{22}(g)(x,\, 0)} = \frac{\alpha_0}{a_{22}}.$$

Then the operator $u \to (Q_0 + \mu^2)u$ *is an isomorphism from* $h^{2,\alpha}(A)$ *onto* $h^{1,\alpha}(A)$*. Moreover, the following estimates hold*:

$$\begin{gathered} \sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_1^i} \right\|_{h^{1,\alpha}(A)} + \|A_0 u\|_{h^{1,\alpha}(A)} \leqslant C \|(O_0 + \mu^2)u\|_{h^{2,\alpha}(A)}, \\ C_1 \|u\|_{h^{2,\alpha}(A)} \leqslant \|O_0 + \mu_0^2 u\|_{h^{1,\alpha}(A)} \leqslant C_2 \|u\|_{h^{2,\alpha}(A)} \end{gathered} \tag{6.20}$$

for all $u \in h^{2,\alpha}(A)$ and for sufficiently large μ_0, $\mu > 0$. Particularly, the operator O_0 is positive and $-O_0$ is a generator of an analytic semigroup in $h^{1,\alpha}(A)$.

Proof. Indeed, by using Theorems 6.1, 6.3, 6.5, Results 6.3, 6.5, by reasoning as in Lemma 5.4, Theorem 5.6, Corollary 5.7 in [1] and the perturbation results for space $H(h^{2,\alpha}(A),\, h^{1,\alpha}(A))$ we obtain the assertion. □

Let

$$W_t = \left\{ g \in h_+^{2,\alpha},\ \inf_{x \in (-\infty,\infty)} \left[\frac{t}{\nu + g} \frac{\partial}{\partial x_2} \upsilon_g(x,\, 0) + k_g(x) > 0 \right] \right\},$$

$$k_g(x) = \frac{\alpha(g)(x)}{a_{22}(g)(x,\, 0)}, \quad t \in [0,\, 1].$$

Remark 6.7. Suppose that $g \in W_1$. Then

$$w_\pi = w_g(x,\, 0) < \frac{\alpha_0}{a_{22}} \quad \text{for } x \in (-\infty,\, \infty).$$

From Theorem 5.1 we know that there is a constant $M > 0$ such that

$$\|K(g)\|_{B(h^{2,\beta}(\Gamma;H),h^{2,\beta}(\Omega;H))} \leqslant M, \quad \beta \in (0,\, \alpha)$$

for all $g \in h_+^{2,\beta}$ satisfying $\|g\|_{h^{2,\beta}} \leqslant \chi$. Now define

$$k = \frac{(\eta - \chi)^3 \chi}{2M[1 + (\eta + \chi)^2 + \chi^2](1 + \chi)^2}.$$

7. Coercive Estimates for the Linearization

It is clear that $O_0 = O_{10} + O_{20} + O_{30}$ may be viewed as a principal part of the linearization of $\partial O(g)$ with coefficients fixed in $(x^0,\, 0)$. Our main goal in this section is to prove that the operator O belongs to the class

$$H(h^{2,\alpha}(A),\, h^{1,\alpha}(A)).$$

We use the estimates of local operators in the preceding section to derive coercive estimates for the linearization operator $\partial O(g)$. Let

$$\partial O(g) = O_1(g) + O_2(g) + O_3(g),$$

where

$$\begin{gathered} O_1(g) = B_0(g)K(g), \quad O_2(g) = \partial B_0(g)[.,\, K(g)g], \\ O_3(g) = -B_0(g)S(g)\partial B(g)[.,\, K(g)g]. \end{gathered} \tag{7.1}$$

Given $g \in h_+^{2,\alpha}$ and $t \in [0,\, 1]$, set

$$\partial O^t(g) = O_1(g) + t\partial B_0(g)[.,\, K(g)g] - tB_0(g)S(g)\partial B(g)[.,\, K(g)g]$$

and observe that

$$\partial O^1(g) = \partial O(g).$$

Let $\delta > 0$ be given and let $\{V_j,\, \varphi_j,\, j \in \mathbb{N}\}$ denote δ-localizasion sequence for $S = \mathbb{R} \times (-\frac{\delta}{2},\, \frac{\delta}{2})$, the covering $\{V_j,\, j \in \mathbb{N}\}$ has finite multiplicity, diam $U_j < \delta$, and $\{V_j,\, \varphi_j,\, j \in \mathbb{N}\}$ is a partition of unity on S with $\sum_{j \in \mathbb{N}} \varphi_j(x) \equiv 1$. Moreover, we fix $x_{1j} \in \mathbb{R}$ such that $(x_{1j},\, 0) \in V_j$, $j \in \mathbb{N}$.

Here, we will prove the following result:

Theorem 7.1. *Assume that Conditions* 3.1 *and* 3.2 *are satisfied. Suppose that* $W_0 \subset W_1$ *is compact,* $\beta \in (0, \alpha)$ *and that* $k > 0$. *Then there exist* $\delta \in (0, 1]$, *a* δ*-localization sequence* $\{V_j, \varphi_j, j \in \mathbb{N}\}$, $\beta \in (0, \alpha)$ *and a positive constant* $C = C(W_0, M, \delta)$ *such that*

$$\left\|[\varphi_j \partial O_t(g) - O_\pi(g, x_{1j})\varphi_j]\upsilon\right\|_{h^{1,\alpha}(A)} \leqslant k\|\varphi_j\upsilon\|_{h^{2,\alpha}(A)} + C\|\upsilon\|_{h^{2,\beta}(A)}$$

for all $\upsilon \in h^{2,\alpha}(A)$, $j \in \mathbb{N}$, $t \in [0, 1]$ *and* $g \in W_0$.

For proving Theorem 7.1 we need some preparation. Let

$$A_j = \partial A(g)(x_{1j}, 0).$$

Consider the equation

$$(O(g) + \mu)u = f \tag{7.2}$$

for

$$f = [O(g) + \mu]u \in B_{1p}.$$

From (7.2) for $u_j = u\varphi_j$, $u \in D_A(\alpha)$ we get

$$(O(g) + \mu)u_j = \sum_{k=1}^{3} O_k(g)u_j + \mu u_j = \sum_{k=1}^{3} F_{kj} + f.\varphi_j, \tag{7.3}$$

where

$$\begin{aligned}
F_{1j} &= \gamma\Big[\Big(b_1\frac{\partial\varphi_j}{\partial x_1} + b_2\frac{\partial\varphi_j}{\partial x_2}\Big)U_1(g)\Big], \qquad U_1(g) = K(g)u,\\
F_{2j} &= \gamma\Big[-\frac{\partial\upsilon}{\partial x_1}\frac{\partial\varphi_j}{\partial x_1} + \frac{2g_{x_1}}{\nu + g}\frac{\partial\upsilon}{\partial x_2}\frac{\partial\varphi_j}{\partial x_1}\Big]u, \qquad \upsilon = \upsilon(g) = K(g)g,\\
F_{3j} &= \gamma\Big[\Big(b_1\frac{\partial\varphi_j}{\partial x_1} + b_2\frac{\partial\varphi_j}{\partial x_2}\Big)U_3(g)\Big], \qquad U_3(g) = S(g)\partial B(g)[u, K(g)g].
\end{aligned} \tag{7.4}$$

By freezing in (7.3) coefficients at points $(x_{1j}, 0)$ we have localized equations

$$(O(x_{1j}) + \mu)u_j = F_j, \tag{7.5}$$

where

$$\begin{gathered}
F_j = \sum_{k=1}^{3} F_{kj} + f.\varphi_j + \sum_{k=1}^{3}(O_{kj} - O_j)u_j,\\
(O_{1j} - O_1)u_j = B_0(g)[K_j - K(g)]u_j + [B_{0j} - B_0(g)]K_j u_j,\\
(O_{2j} - O_2)u_j = -\Big(\frac{\partial\upsilon_j}{\partial x_1} - \frac{\partial\upsilon}{\partial x_1}\Big)\frac{\partial}{\partial x_1}u_j + \Big(\frac{\partial\upsilon_j}{\partial x_2} - \frac{\partial\upsilon}{\partial x_2}\Big)(\Lambda_1 + \Lambda_2)u_j,\\
(O_{3j} - O_3)u_j = \sum_{i=1}^{4}(O_{3ij} - O_{3i})u_j;
\end{gathered} \tag{7.6}$$

here,

$$O_{3ij} = B_0(g)S(g)G_i(g)(x_{1j}, 0),$$

and O_{3ij}, G_i are defined as in (6.14); moreover,

$$O(x_{1j}) = O(x_{1j},\, 0) = \sum_{k=1}^{3} O_k(g)(x_{1j}), \quad O_k(x_{1j}) = O_k(g)(x_{1j},\, 0),$$
$$B_{0j} = B_0(g)(x_{1j},\, 0), \quad K_j = K(g)(x_{1j},\, 0), \quad \upsilon_j = \upsilon_j(g) = K(g)(x_{1j},\, 0)g,$$
$$S_j = S(g)\partial B(g)[.,\, K(g)g](x_{1j},\, 0),$$

$O_k(x_{1j})$ are local operators fixed at points $(x_{1j},\, 0)$ defined by equalities (6.3), (6.5), (6.14), (6.15), respectively. From expressions of F_j, $K(g)(x_{1j},\, 0)$ by using (6.3), (6.5), (6.14), (6.15) we get that $F_j \in h^{1,\alpha}(A)$.

For proving Theorem 7.1 we need the following lemmas.

Lemma 7.1. *The operator $u \to (O(x_{1j}) + \mu^2)u$ is an isomorphism from $h^{2,\alpha}(A)$ onto $h^{1,\alpha}(A)$. Moreover, the following estimate holds*:

$$\sum_{i=0}^{1} |\mu|^{2-i} \left\| \frac{\partial^i u}{\partial x_1^i} \right\|_{h^{1,\alpha}(A)} + \|A_j u\|_{h^{1,\alpha}(A)} \leqslant C \|(O(x_{1j}) + \mu^2)u\|_{h^{1,\alpha}(A)} \tag{7.7}$$

for all $u \in h^{2,\alpha}(A)$ and for sufficiently large $\mu > 0$.

Proof. Consider the equation

$$(O(x_{1j}) + \mu^2)u = f.$$

Then, by virtue of Theorem 6.6 we obtain that the operator $u \to (O(x_{1j}) + \mu^2)u$ is an isomorphism from $h^{2,\alpha}(A)$ onto $h^{1,\alpha}(A)$ and the estimate (7.7) holds. □

Lemma 7.2. *There is a positive $\varepsilon \in (0,\, 1)$ such that the following local estimate holds*:

$$\|(O_{1j} - O_1)u_j\|_{h^{1,\alpha}(A)} \leqslant \varepsilon \|K_j u_j\|_{h^{1,\alpha}(A)}. \tag{7.8}$$

Proof. From (7.6) we get

$$\|(O_{1j} - O_1)u_j\|_{h^{1,\alpha}(A)} \leqslant \|B_0(g)[K_j - K(g)]u_j\|_{h^{1,\alpha}(A)} + \|[B_{0j} - B_0(g)]K_j u_j\|_{h^{1,\alpha}(A)}.$$

Then considering the expressions $B_0(g)$, B_{0j}, $K(g)$, K_j and in view of smoothness of coefficients, choosing δ sufficiently small, we have

$$\begin{aligned}
\|(O_{1j} - O_1)u_j\|_{h^{1,\alpha}(A)} &\leqslant \|B_0(g)[K_j - K(g)]u_j\|_{h^{1,\alpha}(A)} \\
&\quad + \left\|(b_1 - b_{1j})\frac{\partial}{\partial x_1}K_j u_j\right\|_{h^{1,\alpha}(A)} + \left\|(b_2 - b_{2j})\frac{\partial}{\partial x_2}K_j u_j\right\|_{h^{1,\alpha}(A)} \\
&\leqslant \varepsilon\left(\left\|\frac{\partial}{\partial x_1}K_j u_j\right\|_{h^{1,\alpha}(A)} + \left\|\frac{\partial}{\partial x_2}K_j u_j\right\|_{h^{1,\alpha}(A)}\right) \leqslant \varepsilon \|K_j u_j\|_{h^{2,\alpha}(A)}. \quad \square
\end{aligned}$$

Lemma 7.3. *There is a positive $\varepsilon \in (0,\, 1)$ such that the following local estimate holds*:

$$\|(O_{2j} - O_2)u_j\|_{h^{1,\alpha}(A)} \leqslant \varepsilon \|u_j\|_{h^{1,\alpha}(A)}. \tag{7.9}$$

Proof. From the (7.6), we get

$$\|(O_{2j}-O_2)u_j\|_{h^{1,\alpha}(A)} \leqslant \left\|\Big(\frac{\partial \upsilon_j}{\partial x_1}-\frac{\partial \upsilon}{\partial x_1}\Big)\frac{\partial}{\partial x_1}u_j\right\|_{h^{1,\alpha}(A)} + \left\|\Big(\frac{\partial \upsilon_j}{\partial x_2}-\frac{\partial \upsilon}{\partial x_2}\Big)(\Lambda_1+\Lambda_2)u_j\right\|_{h^{1,\alpha}(A)}.$$

Then, by using the smoothness of coefficients and choosing δ sufficiently small, we get the estimate (7.9). □

Lemma 7.4. *There is a positive $\varepsilon \in (0, 1)$ such that the following local estimate holds:*

$$\|(O_{3j}-O_3)u_j\|_{h^{1,\alpha}(A)} \leqslant \varepsilon\|u_j\|_{h^{1,\alpha}(A)}. \tag{7.10}$$

Proof. From expressions (6.13)–(6.14) we get

$$\|(O_{3j}-O_3)u_j\|_{h^{1,\alpha}(A)} \leqslant \sum_{i=1}^{4}\|(O_{3ij}-O_{3i})u_j\|_{h^{1,\alpha}(A)}. \tag{7.11}$$

Moreover, from expressions O_{31} and G_1 in (6.14) by boundedness of functions g, ν, β we have

$$\begin{aligned}
&\|(O_{31j}-O_{31})u_j\|_{h^{1,\alpha}(A)}\\
&\quad\leqslant C\left\|(B_0(g)S(g)(x_{1j},0)-B_0(g)S(g)(x))\Big(u_j+\frac{\partial u_j}{\partial x_1}\Big)\right\|_{h^{1,\alpha}(A)}\\
&\quad\leqslant C\bigg\{\big\|[B_0(g)S(g)(x_{1j},0)-B_0(g)S(g)(x)]u_j\big\|_{h^{1,\alpha}(A)}\\
&\qquad+\big\|B_0(g)S(g)(x)[u_j(x)-u_j(x_{1j},0)]\big\|_{h^{1,\alpha}(A)}\\
&\qquad+\left\|[B_0(g)S(g)(x_{1j},0)-B_0(g)S(g)(x)]\frac{\partial u_j}{\partial x_1}\right\|_{h^{1,\alpha}(A)}\\
&\qquad+\left\|B_0(g)S(g)(x)\left[\frac{\partial u_j}{\partial x_1}-\frac{\partial}{\partial x_1}u_j(x_{1j},0)\right]\right\|_{h^{1,\alpha}(A)}\bigg\}.
\end{aligned}$$

Then by boundedness of operator $B_0(g)S(g)$, smoothness of coefficients, choosing δ appropriately, we obtain

$$\|(O_{31j}-O_{31})u_j\|_{h^{1,\alpha}(A)} \leqslant \varepsilon\|u_j\|_{h^{2,\alpha}(A)}. \tag{7.12}$$

In a similar way, from expressions O_{32} and G_2 in (6.14) we have

$$\|(O_{32j}-O_{32})u_j\|_{h^{1,\alpha}(A)} \leqslant \varepsilon\|u_j\|_{h^{2,\alpha}(A)}. \tag{7.13}$$

In view of the condition on the operator $\partial A(g)$, boundedness of the operator $B_0(g)S(g)$, smoothness of coefficients, choosing δ appropriately we obtain

$$\|(O_{33j}-O_{33})u_j\|_{D_A(\alpha)} \leqslant \varepsilon\|u_j\|_{D_A(\alpha)}. \tag{7.14}$$

Finally, from expressions O_{34} and G_4 in (6.14) by boundedness of functions g, ν, β, we have

$$\begin{aligned}
&\|(O_{34j} - O_{34})u_j\|_{h^{1,\alpha}(A)} \\
&\leqslant C\left\|(B_0(g)S(g)(x_{1j},\,0) - B_0(g)S(g)(x))\left(u_j + \frac{\partial u_j}{\partial x_1} + \frac{\partial^2 u_j}{\partial x_1^2}\right)\right\|_{h^{1,\alpha}(A)} \\
&\leqslant C\Big\{\left\|[B_0(g)S(g)(x_{1j},\,0) - B_0(g)S(g)(x)]u_j\right\|_{h^{1,\alpha}(A)} \\
&\quad + \left\|B_0(g)S(g)(x)[u_j(x) - u_j(x_{1j},\,0)]\right\|_{h^{1,\alpha}(A)} \\
&\quad + \left\|[B_0(g)S(g)(x_{1j},\,0) - B_0(g)S(g)(x)]\frac{\partial u_j}{\partial x_1}\right\|_{h^{1,\alpha}(A)} \\
&\quad + \left\|B_0(g)S(g)(x)\left[\frac{\partial u_j}{\partial x_1} - \frac{\partial}{\partial x_1}u_j(x_{1j},\,0)\right]\right\|_{h^{1,\alpha}(A)} \\
&\quad + \left\|[B_0(g)S(g)(x_{10},\,0) - B_0(g)S(g)(x)]\frac{\partial^2 u_j}{\partial x_1^2}\right\|_{D_A(\alpha)} \\
&\quad + \left\|B_0(g)S(g)(x)\left[\frac{\partial^2 u_j}{\partial x_1^2} - \frac{\partial^2}{\partial x_1^2}u_j(x_{1j},\,0)\right]\right\|_{B_{1p}} \leqslant \varepsilon\|u_j\|_{h^{1,\alpha}(A)}\Big\}. \qquad (7.15)
\end{aligned}$$

Then the estimate (7.10) is obtained from (7.12)–(7.15). □

Now, we can prove Theorem 7.1.

Proof of Theorem 7.1. By virtue of Theorem 6.6 the estimate

$$\sum_{i=0}^{1} |\mu|^{2-i}\left\|\frac{\partial^i u_j}{\partial x_1^i}\right\|_{h^{1,\alpha}(A)} + \|A_0 u_j\|_{h^{1,\alpha}(A)} \leqslant C\|F_j\|_{h^{1,\alpha}(A)} \qquad (7.16)$$

holds for any solution $u_j \in h^{2,\alpha}(A)$ of the equation (7.5).

Whence, using smoothness of coefficients of equations (7.3)–(7.6), in view of Lemmas 7.1–7.3 for μ with sufficiently large $\operatorname{Re}\mu > 0$, we get

$$\|F_j\|_{h^{1,\alpha}(A)} \leqslant \varepsilon\left[\sum_{i=0}^{1} |\mu|^{2-i}\left\|\frac{\partial^i u_j}{\partial x_1^i}\right\|_{h^{1,\alpha}(A)} + \|A_j u_j\|_{h^{1,\alpha}(A)}\right] + \|f.\varphi_j\|_{h^{1,\alpha}(A)}. \qquad (7.17)$$

Moreover, by applying the microlocal analysis in a similar way as in [9, Lemma 6.5–6.7] we can obtain the same estimates for corresponding commutators operators. Then from this and from (7.16), (7.17), we get the assertion. □

From Theorem 7.1, we have

Result 7.1. Assume that Conditions 3.1, 3.2 are satisfied and $K \subset W$ is compact. Then there exist positive constants μ_0 and $C = C(K)$ such that

$$\|\upsilon\|_{h^{2,\alpha}(A)} + |\mu|\|\upsilon\|_{h^{1,\alpha}(A)} \leqslant C\|(\mu + \partial O_t(g))\upsilon\|_{h^{1,\alpha}(A)}$$

for all $\upsilon \in h^{2,\alpha}(A)$, $g \in K$, $t \in [0,\,1]$ and $\mu \in \{z\colon \operatorname{Re} z > \mu_0\}$.

Now, by reasoning as in [9, §6, p. 32] and by using Theorems 6.1–6.6, 7.1, we obtain the assertions of Theorem 1.

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