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NONLOCAL SEPARABLE ELLIPTIC EQUATIONS AND APPLICATIONS

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Abstract. The regularity properties of nonlocal elliptic equations are investigated in abstract weighted L_p spaces. Here, we find sufficient conditions that guarantee the separability of the linear problems. We prove that the corresponding nonlocal elliptic operator is sectorial and is also a negative generator of an analytic semigroup. In application, the maximal regularity properties of the for degenerate abstract equation in $L_\mathbf{p}$ norms, and infinite systems of degenerate elliptic integro-differential equations with parameters are obtained.

Keywords: sectorial operators, abstract weighted spaces, operator-valued multipliers, nonlocal equations, degenerate integro-differential equations.

AMS Subject Classification: 35Axx, 35Jxx, 35Kxx, 35J70.

1. Introduction, notations and background

In recent years, regularity properties of abstract differential equations, especially elliptic and parabolic type have been studied extensively in [1, 3, 5, 6, 11, 12, 16-19, 23] and the references therein. Moreover, nonlocal (or convolution) differential equations have been treated in [3-10, 14-18, 19, 21] (for comprehensive references see [14, 15, 21]). Convolution operators in Banach-valued function spaces studied in [3, 12, 17, 18, 20-23]. However, the nonlocal differential operator equations are relatively less investigated subject. In [13, 17, 18] the regularity properties of degenerate nonlocal differential operator equations are studied.

The concept of non-locality in equations considered here, because of our equations involve the convolution terms, i.e. integral terms that connects the values of the solution at various points with each other.

The main aim of the present paper is to study the maximal regularity properties of the linear nonlocal differential operator equations with parameters

$$\sum_{|\alpha|\leq l} \varepsilon_\alpha a_\alpha * D^\alpha u + A * u + \lambda u = f(x),\ x \in \mathbb{R}^n \tag{1}$$

in weighted $L_{p,\gamma}$ spaces, where $a_\alpha = a_\alpha(x)$ are complex-valued functions, l is a natural number, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, α_k are nonnegative integers, $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$, $\varepsilon_\alpha = \prod_{k=1}^{n} \varepsilon_k^{\frac{\alpha_k}{l}}$, ε_k are positive, λ is a complex parameter and $A = A(x)$ is a linear operator in a Banach space E for $x \in \mathbb{R}^n$.

In [13] we studied the regularity properties of the linear equations (1), when $\varepsilon_1 = \varepsilon_2 =, ..., = \varepsilon_n = 1$.

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In the applications, particularly the above equations describe the charged particle motion for certain configurations of oscillating magnetic fields. Moreover, a number of nonlocal continuum models have been proposed in order to understand aggregation in biological systems, see [8] and references therein. Several of such models lead to nonlocal equations with degenerate diffusion. Maximal regularity has proven very useful in handling some concrete non-linear evolution equations as shown by the papers [2] and [14] which deal with the Navier-Stokes equations of fluid dynamics. One of main features of the present work is that the nonlocal equations are degenerate on some points of $\mathbb{R}=(-\infty,\infty)$ and the equation (1) has a variable operator coefficient. Moreover, we prove that the operator generated by problem (1) is $\varphi-$ sectorial. The main tools of this work is the theory of operator-valued Fourier multipliers. Since the equation (1) has an unbounded operator coefficient, some difficulties occur. This fact is derived by using the representation formula for the solution of (1) and operator valued multipliers in $L_{p,\gamma}\left(\mathbb{R}^n;E\right)$.

Let E be a Banach space and $\gamma=\gamma(x)$, $x=(x_1,x_2,...,x_n)$ be a positive measurable weighted function on a measurable subset $\Omega\subset\mathbb{R}^n$. Let $L_{p,\gamma}(\Omega;E)$ denote the space of strongly $E-$valued functions that are defined on Ω with the norm

$$\|f\|_{L_{p,\gamma}}=\|f\|_{L_{p,\gamma}(\Omega;E)}=\left(\int\limits_{\Omega}\|f(x)\|_E^p\,\gamma(x)dx\right)^{1/p},\ 1\le p<\infty,$$

$$\|f\|_{L_{\infty,\gamma}(\Omega;E)}=ess\sup_{x\in\Omega}\left[\gamma(x)\,\|f(x)\|_E\right].$$

For $\gamma\left(x\right)\equiv1$, the space $L_{p,\gamma}(\Omega,E)$ will be denoted by $L_p=L_p(\Omega;E)$. The weight $\gamma=\gamma\left(x\right)$ satisfy an A_p condition, i.e., $\gamma\in A_p$, $p\in(1,\infty)$ if there is a positive constant C such that

$$\sup_Q\left(\frac{1}{|Q|}\int\limits_Q\gamma(x)dx\right)\left(\frac{1}{|Q|}\int\limits_Q\gamma^{-\dfrac{1}{p-1}}(x)dx\right)^{p-1}\le C$$

for all cubes $Q\subset\mathbb{R}^n$ (see $[10,Ch.9]$).

Here, $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{R}$ denotes the set of real numbers. Let $\mathbb{C}$ be the set of complex numbers and

$$S_\varphi=\left\{\lambda\in\mathbb{C},\quad|\arg\lambda|\le\varphi\right\}\cup\{0\},\quad0\le\varphi<\pi.$$

Let E_1 and E_2 be two Banach spaces and let $B\left(E_1,E_2\right)$ denote the space of bounded linear operators from E_1 to E_2. For $E_1=E_2=E$ we denote $B\left(E,E\right)$ by $B\left(E\right)$.

Let A be a linear operator in E. Assume that $D\left(A\right)$, $R\left(A\right)$ denote the domain and range of the linear operator in E, respectively. Let Ker A denote a null space of A.

A closed linear operator A is said to be $\varphi-$ sectorial (or sectorial for $\varphi=0$) in a Banach space E with bound $M>0$ if Ker $A=\{0\}$, $D\left(A\right)$ and $R\left(A\right)$ are dense on E, and $\left\|\left(A+\lambda I\right)^{-1}\right\|_{B(E)}\le M\left|\lambda\right|^{-1}$ for all $\lambda\in S_\varphi$, $\varphi\in[0,\ \pi)$, where I is an identity operator in E. Sometimes $A+\lambda I$ will be written as $A+\lambda$ and will be denoted by A_λ. It is known (see [22]) that the fractional powers of the operator A are well defined. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graph norm

$$\|u\|_{E(A^\theta)}=\left(\|u\|_E^p+\left\|A^\theta u\right\|_E^p\right)^{\frac{1}{p}},\ \ 1\le p<\infty,\ -\infty<\theta<\infty.$$

Note that the above norms are equivalent for $p\in[1,\infty)$. Here, $S=S(\mathbb{R}^n;E)$ denotes the $E-$valued Schwartz class, i.e. the space of $E-$valued rapidly decreasing smooth functions on

$\mathbb{R}^n$, equipped with its usual topology generated by seminorms. $S(\mathbb{R}^n;\mathbb{C})$ will be denoted by just S.

Let $S'(\mathbb{R}^n;E)$ denote the space of all continuous linear operators, $L : S \to E$, equipped with topology of bounded convergence. Recall $S(\mathbb{R}^n;E)$ is norm dense in $L_{p,\gamma}(\mathbb{R}^n;E)$ when $1 < p < \infty$, $\gamma \in A_p$.

Here, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, where α_i are integers. An $E-$ valued generalized function $D^\alpha f$ is called a generalized derivative in the sense of Schwartz distributions of the function $f \in S(\mathbb{R}^n;E)$ if

$$\langle D^\alpha f, \varphi\rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi\rangle$$

holds for all $\varphi \in S$.

Let F denotes the Fourier transform defined by

$$\hat{u}(\xi) = Fu = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-ix\xi} u(x)\, dx \text{ for } u \in S(\mathbb{R}^n;E) \text{ and } x,\, \xi \in \mathbb{R}^n.$$

Throughout this section the Fourier transformation of a function f will be denoted by $\widehat{f}$ and $F^{-1}f = \check{f}$. It is known that

$$F(D_x^\alpha f) = (i\xi_1)^{\alpha_1} ... (i\xi_n)^{\alpha_n} \widehat{f}, \quad D_\xi^\alpha (F(f)) = F\left[(-ix_1)^{\alpha_1} ... (-ix_n)^{\alpha_n} f\right]$$

for all $f \in S'(\mathbb{R}^n;E)$.

The equation (1) is a non-degenerated equation because of these equation does not include weighted function (i.e. the functon $\gamma(x)$ approaches to 0 or ∞). But, the equations (10) and (13) are degenerated equations due to it contain term degenerated term $D^{[\alpha]}u$, where

$$D^{[\alpha]} = D_{x_1}^{[\alpha_1]} D_{x_2}^{[\alpha_2]} ... D_{x_n}^{[\alpha_n]},\ D_{x_i}^{[\alpha_i]} = \left(\gamma_k(x_k) \frac{\partial}{\partial x_k}\right)^{\alpha_k},\ k = 1, 2, ..., n.$$

Suppose that E_1 and E_2 are two Banach spaces. A function $\Psi \in L_\infty(\mathbb{R}^n; B(E_1,E_2))$ is called a Fourier multiplier from $L_{p,\gamma}(\mathbb{R}^n;E_1)$ to $L_{p,\gamma}(\mathbb{R}^n;E_2)$ for $p \in (1,\infty)$ if the map $u \to Tu = F^{-1}\Psi(\xi)Fu$, $u \in S(\mathbb{R}^n;E_1)$ is well defined and extends to a bounded linear operator

$$T : L_{p,\gamma}(\mathbb{R}^n;E_1) \to L_{p,\gamma}(\mathbb{R}^n;E_2).$$

A Banach space E is called a UMD space (see [8], [23]) if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon\to 0} \int\limits_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is initially defined on $S(\mathbb{R};E)$ and is bounded in $L_p(\mathbb{R};E)$, $p \in (1,\infty)$ (see [6, 18]). UMD spaces include L_p, l_p spaces and Lorentz spaces L_{pq}, $p, q \in (1,\infty)$.

A set $K \subset B(E_1,E_2)$ is called $R-$ bounded (see [5], [21]) if there is a constant $C > 0$ such that for all $T_1, T_2, ..., T_m \in K$ and $u_1, u_2, ..., u_m \in E_1$, $m \in \mathbb{N}$

$$\int\limits_0^1 \left\|\sum_{j=1}^m r_j(y) T_j u_j\right\|_{E_2} dy \le C \int\limits_0^1 \left\|\sum_{j=1}^m r_j(y) u_j\right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1;1\}-$ valued random variables on $[0,1]$. The smallest C for which the above estimate holds is called the $R-$ bound of K and denoted by $R(K)$.

Definition 1.1. *A Banach space E is said to be a space satisfying the multiplier condition with respect to weighted function γ and $p \in (1,\infty)$ (or multiplier condition with respect to $p \in (1,\infty)$ when $\gamma(x) \equiv 1$) if for any $\Psi \in C^{(n)}(\mathbb{R}^n \setminus \{0\}; B(E))$ the $R-$boundedness of the set*

$$\left\{ |\xi|^{|\beta|} D_\xi^\beta \Psi(\xi) : \xi \in \mathbb{R}^n \setminus \{0\},\ \beta = (\beta_1, \beta_2, ..., \beta_n),\ \beta_k \in \{0,1\} \right\}$$

implies that Ψ is a Fourier multiplier in $L_{p,\gamma}(\mathbb{R}^n; E)$, i.e., $\Psi \in M_{p,\gamma}^{p,\gamma}(E)$ (or Ψ is a Fourier multiplier in $L_p(\mathbb{R}^n; E)$, i.e., $\Psi \in M_p^p(E)$).

Remark 1.1. *Note that, if E is UMD space then it satisfies the multiplier condition with respect to $p \in (1,\infty)$ (see* [5], [8], [21]*).*

Definition 1.2. *A sectorial operator $A(x)$, $x \in \mathbb{R}^n$ is said to be uniformly $R-$sectorial in a Banach space E if there exists a $\varphi \in [0, \pi)$ such that*

$$\sup_{x \in \mathbb{R}^n} R\left(\left\{ \left[A(x)(A(x) + \xi I)^{-1} \right] : \xi \in S_\varphi \right\}\right) \leq M.$$

Let $A = A(x)$, $x \in \mathbb{R}^n$ be closed linear operator in E with domain $D(A)$ independent of x. The Fourier transformation of $A(x)$ is a linear operator with the domain $D(A)$ defined as:

$$\hat{A}(\xi) u(\varphi) = A(x) u(\hat{\varphi}) \text{ for } u \in S'(\mathbb{R}^n; E(A)),\ \varphi \in S(\mathbb{R}^n),$$

where $\langle f, \varphi \rangle$ denote the value of generalized function f on the $\varphi \in S(\mathbb{R}^n)$ (for more details [2] and [5]).

Let $A = A(x)$ be a linear operator with domain $D(A)$ independent on $x \in \mathbb{R}^n$ such that $Au \in L^1(\mathbb{R}^n; E)$ for $u \in S(\mathbb{R}^n; D(A))$. The convolution $A * u$ of A and $u \in S(\mathbb{R}^n; D(A))$ is defined as:

$$A * u = \int_{\mathbb{R}^n} A(x) u(x - \xi) d\xi \text{ for } u \in S(\mathbb{R}^n; D(A)).$$

Remark 1.2. *By using the Fourier transform, in a similar way as in the scalar case, we obtain that*

$$F(A * u) = \hat{A}(\xi) \hat{u}(\xi) \text{ for } u \in S(\mathbb{R}^n; D(A)), \text{ here } \hat{A}(\xi) = (FA)(\xi).$$

Note that, in Hilbert spaces every norm bounded set is $R-$ bounded. Therefore, in Hilbert spaces all sectorial operators are $R-$ sectorial.

The problem (1) is separable in $L_p = L_p(\mathbb{R}^n; E)$ if for all $f \in L_p$ the problem has a unique strong solution $u \in W^l(\mathbb{R}^n; E(A), E)$ and the following uniform coercive estimate holds:

$$\sum_{|\alpha| \leq l} \varepsilon_\alpha \|a_\alpha * D^\alpha u\|_{L_p} + \|A * u\|_{L_p} \leq C \|f\|_{L_p},$$

where a positive constant C does not depent on parameters ε and λ.

Let E_0 and E be two Banach spaces, where E_0 is continuously and densely embedded into E. Let l be a natural number. $W_{p,\gamma}^l(\mathbb{R}^n; E_0, E)$ denotes the space of all functions from $S'(\mathbb{R}^n; E_0)$ such that $u \in L_{p,\gamma}(\mathbb{R}^n; E_0)$ and $D_k^l u \in L_{p,\gamma}(\mathbb{R}^n; E)$ with the norm

$$\|u\|_{W_{p,\gamma}^l(\mathbb{R}^n; E_0, E)} = \|u\|_{L_{p,\gamma}(\mathbb{R}^n; E_0)} + \sum_{|\alpha| \leq l} \|D^\alpha u\|_{L_{p,\gamma}(\mathbb{R}^n; E)} < \infty.$$

Similarly, it is clear that

$$\|u\|_{W_{p,\gamma}^{[l]}(\mathbb{R}^n; E_0, E)} = \|u\|_{L_p(\mathbb{R}^n; E_0)} + \sum_{|\alpha| \leq l} \left\| D^{[\alpha]} u \right\|_{L_p(\mathbb{R}^n; E)} < \infty.$$

In a similar way in [5, Theorem 3.25], we obtain:

Proposition 1.1. *Let E be a UMD space and $\gamma \in A_p$. Assume that Ψ_h is a set of operator functions in $C^n\left(\mathbb{R}^n \backslash \{0\}; B(E)\right)$ depending on the parameter $h \in Q \in R$ and there exists a positive constant K such that*

$$\sup_{h \in Q} R\left(\left\{|\xi|^{|\beta|} D^\beta \Psi_h(\xi) : \xi \in \mathbb{R}^n \backslash \{0\},\ \beta_k \in \{0,1\}\right\}\right) \leq K.$$

Then, the set Ψ_h is a uniformly bounded collection of the Fourier multipliers in $L_{p,\gamma}(\mathbb{R}^n; E)$.

Let E_1 and E_2 be two Banach spaces. Suppose that $T \in B(E_1, E_2)$ and $1 \leq p < \infty$. Then, $\tilde{T} \in B\left(L_{p,\gamma}(\mathbb{R}^n; E_1), L_{p,\gamma}(\mathbb{R}^n; E_2)\right)$ will denote an operator $\left(\tilde{T} f\right)(x) = T(f(x))$ for $f \in L_{p,\gamma}(\mathbb{R}^n; E_1)$ and $x \in \mathbb{R}^n$.

In a similar way as in [5], [16], we have

Proposition 1.2. *Let $1 \leq p < \infty$, $\gamma \in A_p$. If $W \in B(E_1, E_2)$ is $R-$bounded, then the collection $\tilde{W} = \left\{\tilde{T} : T \in W\right\} \subset B\left(L_{p,\gamma}(\mathbb{R}^n; E_1), L_{p,\gamma}(\mathbb{R}^n; E_2)\right)$ is also $R-$ bounded.*

2. Nonlocal separable elliptic equation

In the our old work (see [13], [18]), it was proved that the operator functions of the

$$\sigma_{0\varepsilon}(\xi, \lambda) = \lambda D_\varepsilon(\xi, \lambda), \sigma_{1\varepsilon}(\xi, \lambda) = \hat{A}(\xi) D_\varepsilon(\xi, \lambda),$$

$$\sigma_{2\varepsilon}(\xi, \lambda) = \sum_{|\alpha| \leq l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \hat{a}_\alpha(\xi) (i\xi)^\alpha D_\varepsilon(\xi, \lambda),\ |\xi|^{|\beta|} D_\xi^\beta \sigma_{i\varepsilon}(\xi, \lambda), i = 0, 1, 2 \tag{2}$$

are uniformly bounded and sets of the $S_{i\varepsilon}(\xi, \lambda) = \left\{|\xi|^{|\beta|} D_\xi^\beta \sigma_{i\varepsilon}(\xi, \lambda);\ \xi \in \mathbb{R}^n \backslash \{0\}\right\}$, are uniformly $R-$ bounded for $\beta_k \in \{0,1\}$ and $0 \leq |\beta| \leq n$, where $D_\varepsilon(\xi, \lambda) = \left[\hat{A}(\xi) + L_\varepsilon(\xi) + \lambda\right]^{-1}$, $L_\varepsilon(\xi) = \sum\limits_{|\alpha| \leq l} \varepsilon_\alpha \hat{a}_\alpha(\xi) (i\xi)^\alpha$.

Now, we are ready to present our main results. Consider the following nondegenerate nonlocal differential operator equations

$$\sum_{|\alpha| \leq l} \varepsilon_\alpha a_\alpha * D^\alpha u + (A + \lambda) * u = f, \tag{3}$$

where ε, ε_α, λ are parameters, a_α are complex-valued functions defined in (1) and A is a linear operator in a Banach space E.

Condition 2.1. *Suppose that the following are satisfied:*

(1) $L_\varepsilon(\xi) = \sum\limits_{|\alpha| \leq l} \varepsilon_\alpha \hat{a}_\alpha(\xi) (i\xi)^\alpha \in S_{\varphi_1}$, $\varphi_1 \in [0,\ \pi)$ for $\xi \in \mathbb{R}^n$,

$|L_\varepsilon(\xi)| \geq C \sum\limits_{k=1}^{n} \varepsilon_k \left|\hat{a}_{\alpha(l,k)}\right| |\xi_k|^l$, $\alpha(l,k) = (0, 0, ..., l, 0, 0, ..., 0)$, i.e $\alpha_i = 0$, $i \neq k$, $\alpha_k = l$;

(2) $\hat{a}_\alpha \in C^{(n)}(\mathbb{R}^n)$ *and*

$$|\xi|^{|\beta|} \left|D^\beta \hat{a}_\alpha(\xi)\right| \leq C_1,\ \beta_k \in \{0,1\}, 0 \leq |\beta| \leq n;$$

(3) *for* $0 \leq |\beta| \leq n$, ξ, $\xi_0 \in \mathbb{R}^n \backslash \{0\}$:

$$\left[D^\beta \hat{A}(\xi)\right] \hat{A}^{-1}(\xi_0) \in C(\mathbb{R}^n; B(E)),\ |\xi|^{|\beta|} \left\|\left[D^\beta \hat{A}(\xi)\right] \hat{A}^{-1}(\xi_0)\right\|_{B(E)} \leq C_2.$$

Let

$$X = L_{p,\gamma}\left(\mathbb{R}^n; E\right),\ Y = W^l_{p,\gamma}\left(\mathbb{R}^n; E\left(A\right), E\right),\ p \in \left(1, \infty\right).$$

Theorem 2.1. *Assume that Condition 2.1 holds and E is a Banach space satisfying the multiplier condition with respect to weighted function γ and $p \in (1, \infty)$. Let $\hat{A}$ be a uniformly $R-$sectorial operator in E with $\varphi \in [0,\ \pi)$, $\lambda \in S_{\varphi_2}$ and $0 \leq \varphi + \varphi_1 + \varphi_2 < \pi$. Then, problem (2.2) has a unique solution u and the coercive uniform estimate holds*

$$\sum_{|\alpha| \leq l} \varepsilon_\alpha \left|\lambda\right|^{1-\frac{|\alpha|}{l}} \left\|a_\alpha * D^\alpha u\right\|_X + \left\|A * u\right\|_X + \left|\lambda\right| \left\|u\right\|_X \leq C \left\|f\right\|_X \tag{4}$$

for all $f \in X$ and $\lambda \in S_\varphi$.

Proof. By applying to the Fourier transform to equation (3), we get

$$\hat{u}\left(\xi\right) = D_\varepsilon\left(\xi, \lambda\right) \hat{f}\left(\xi\right),\ D_\varepsilon\left(\xi, \lambda\right) = \left[\hat{A}\left(\xi\right) + L_\varepsilon\left(\xi\right) + \lambda\right]^{-1}. \tag{5}$$

Hence, the solution of (3) can be represented as $u\left(x\right) = F^{-1} D_\varepsilon\left(\xi, \lambda\right) \hat{f}$ and there are positive constants C_1 and C_2 such that

$$\begin{gathered} C_1 \left|\lambda\right| \left\|u\right\|_X \leq \left\|F^{-1}\left[\sigma_{0\varepsilon}\left(\xi, \lambda\right) \hat{f}\right]\right\|_X \leq C_2 \left|\lambda\right| \left\|u\right\|_X, \\ C_1 \left\|A * u\right\|_X \leq \left\|F^{-1}\left[\sigma_{1\varepsilon}\left(\xi, \lambda\right) \hat{f}\right]\right\|_X \leq C_2 \left\|A * u\right\|_X, \\ C_1 \sum_{|\alpha| \leq l} \varepsilon_\alpha \left|\lambda\right|^{1-\frac{|\alpha|}{l}} \left\|a_\alpha * D^\alpha u\right\|_X \leq \left\|F^{-1}\left[\sigma_{2\varepsilon}\left(\xi, \lambda\right) \hat{f}\right]\right\|_X \leq \\ C_2 \sum_{|\alpha| \leq l} \varepsilon_\alpha \left|\lambda\right|^{1-\frac{|\alpha|}{l}} \left\|a_\alpha * D^\alpha u\right\|_X, \end{gathered} \tag{6}$$

where $\sigma_{i\varepsilon}\left(\xi, \lambda\right)$ are operators defined by (2). Therefore, it is sufficient to show that the operators $\sigma_{i\varepsilon}\left(\xi, \lambda\right)$ are multipliers in X. By the first assumption on space E, these follow from [13] and [18]. Thus, from (6) and (5) we obtain

$$\begin{gathered} \left|\lambda\right| \left\|u\right\|_X \leq C_0 \left\|f\right\|_X,\ \ \left\|A * u\right\|_X \leq C_1 \left\|f\right\|_X, \\ \sum_{|\alpha| \leq l} \left|\lambda\right|^{1-\frac{|\alpha|}{l}} \left\|a_\alpha * D^\alpha u\right\|_X \leq C_2 \left\|f\right\|_X \end{gathered}$$

for all $f \in X$. Hence, we get the assertion.

Let O_ε be an operator in X generated by problem (3) for $\lambda = 0$, i.e.

$$D\left(O_\varepsilon\right) \subset Y,\ O_\varepsilon u = \sum_{|\alpha| \leq l} \varepsilon_\alpha a_\alpha * D^\alpha u + A * u.$$

□

Result 2.1. Theorem 2.1 implies that the operator O_ε is uniformly separable in X, i.e. for all $f \in X$ there is a unique solution $u \in Y$ of the problem (3), all terms of the equation (3) also are from X and there are positive constants C_1 and C_2 so that

$$C_1 \left\|O_\varepsilon u\right\|_X \leq \sum_{|\alpha| \leq l} \varepsilon_\alpha \left\|a_\alpha * D^\alpha u\right\|_X + \left\|A * u\right\|_X \leq C_2 \left\|O_\varepsilon u\right\|_X.$$

Condition 2.2. *assume that $D(A(x)) = D(\hat{A}(\xi))$, $D(\hat{A}(\xi))$ is dense in E and does not depend on ξ; $A\left(x\right)$ is a uniformly $R-$sectorial in E. Moreover, there exist positive constants C_1, C_2 and $\xi_0 \in \mathbb{R}^n$ such that*

$$C_1 \left\| \hat{A}(\xi_0) u \right\|_E \leq \|A(x) u\|_E \leq C_2 \left\| \hat{A}(\xi_0) u \right\|_E$$

for $u \in D(A)$, $x \in \mathbb{R}^n$.

Theorem 2.2. *Assume that the all the conditions of Theorem 2.1 and Condition 2.2 are satisfied. Then, for $f \in X$ and $\lambda \in S_\varphi$, the problem (3) has a unique solution $u \in Y$ and the coercive uniform estimate holds*

$$\sum_{|\alpha| \leq l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \|D^\alpha u\|_X + \|Au\|_X \leq C \|f\|_X .$$

Proof. By applying the Fourier transform, we obtain that of the solution of equation (3) can be represented as

$$\hat{u}(\xi) = D_\varepsilon(\xi, \lambda) \hat{f}(\xi), \; u(x) = F^{-1} D_\varepsilon(\xi, \lambda) \hat{f}(\xi),$$

where

$$D_\varepsilon(\xi, \lambda) = \left[\hat{A}(\xi) + L_\varepsilon(\xi) + \lambda \right]^{-1}.$$

By Condition 2.2, we get

$$\left\| A(x) F^{-1} D_\varepsilon(\xi, \lambda) f^{\hat{}} \right\|_X \leq M \left\| \hat{A}(\xi_0) F^{-1} D_\varepsilon(\xi, \lambda) \hat{f} \right\|_X .$$

Hence, it is sufficient to show that the operator functions

$$\sum_{|\alpha| \leq l} \varepsilon_\alpha |\lambda|^{1-\frac{|\alpha|}{l}} \xi^\alpha D_\varepsilon(\xi, \lambda), \; \hat{A}(\xi_0) D_\varepsilon(\xi, \lambda)$$

are multipliers in X. Indeed, by the part (3) of Condition 2.1 and $R-$sectoriality of $\hat{A}$, these facts are obtained from proved by reasoning as in the proof Theorem 2.1. □

In fact, the coercive estimates in Theorems 2.1 and 2.2 (see the estimate (3) imply the uniqueness of strong solution of $u \in W^l(\mathbb{R}^n; E(A), E)$ to the problem (1).

Condition 2.3. *Let the Condition 2.2 hold and assume that there exist positive constants C_1 and C_2 such that*

$$C_1 \sum_{k=1}^{n} \varepsilon_k \left| \hat{a}_{\alpha(l,k)} \right| |\xi_k|^l \leq \; |L_\varepsilon(\xi)| \leq C_2 \sum_{k=1}^{n} \varepsilon_k \left| \hat{a}_{\alpha(l,k)} \right| |\xi_k|^l, \; \xi \in \mathbb{R}^n$$

for

$$\alpha(l,k) = (0, 0, ..., l, 0, 0, ..., 0), \text{ i.e } \alpha_i = 0, \; i \neq k \text{ and } \alpha_k = l,$$

and there exists $x_0 \in \mathbb{R}^n$ such that

$$\hat{A}(\xi) A^{-1}(x_0) \in L_\infty(\mathbb{R}^n; B(E)), \; \xi, \; x_0 \in \mathbb{R}^n,$$

$$C_1 \|A(x_0) u\| \leq \|A(x) u\| \leq C_2 \|A(x_0) u\|, \; u \in D(A), \; x \in \mathbb{R}^n.$$

Theorem 2.3. *Assume that the all conditions of Theorem 2.2 and Condition 2.3 are satisfied. Then, for $u \in Y$ there are positive constants M_1 and M_2 such that*

$$M_1 \|u\|_Y \leq \|O_\varepsilon u\|_X \leq M_2 \|u\|_Y . \tag{7}$$

Proof. The left part of the above inequality is derived from Theorem 2.2. So, it remains to prove the right hand side of the estimation. Indeed, from Condition 2.3 for $u \in Y$, we have

$$\|A * u\|_X \leq M \left\| F^{-1} \hat{A} \hat{u} \right\|_X \leq C \left\| F^{-1} \hat{A} A^{-1}(x_0) A(x_0) \hat{u} \right\|_X$$
$$\leq C \left\| F^{-1} A(x_0) \hat{u} \right\|_X \leq C \|Au\|_X . \tag{8}$$

Hence, applying the Fourier transform to equation (3) and by reasoning as Theorem 2.2, it is sufficient to prove that the function $\sum_{|\alpha| \leq l} \varepsilon_\alpha \hat{a}_\alpha \xi^\alpha \left(\sum_{k=1}^n \xi_k^{l_k} \right)^{-1}$ is a uniformly multiplier in X. By the Fourier multipliers theory, it is implies that

$$\sum_{|\alpha| \leq l} \varepsilon_\alpha \|a_\alpha * D^\alpha u\|_X \leq C_1 \left(\|u\|_X + \sum_{|\alpha| \leq l} \|D^\alpha u\|_X \right) \tag{9}$$

for all $u \in Y$.

Then, by definition of the space Y, from (8) and (9) we get the estimation (7). □

Consider the following example.

Example 2.1. Let $m = 2$, $n = 2$, $E = \mathbb{C}$, $a_{(2,0)} = a_{11}(x, y)$, $a_{(1,2)} = a_{12}(x, y)$, $a_{(2,0)} = a_{22}(x, y)$, $A = b(x, y)$ such that $\hat{a}_{ij}(\xi)$ and $\hat{b}(\xi)$, $\xi = (\xi_1, \xi_2)$ are positive real valued functions for all $\xi \in \mathbb{R}^2$ satisfying the Condition 2.1.

Consider the parameter dependent nonlocal equation such that:

$$-\varepsilon_1 a_{11} * D_x^2 u - \varepsilon_1^{\frac{1}{2}} \varepsilon_2^{\frac{1}{2}} a_{12} * D_x D_y u - \varepsilon_2 a_{22} * D_y^2 u + b * u = f(x, y) .$$

Then by Theorem 2.1, the above problem is uniform $L_{p,\gamma}(\mathbb{R}^2)$ separable.

3. Degenerate nonlocal elliptic equations

We consider now the following degenerate nonlocal equation

$$\sum_{|\alpha| \leq l} \varepsilon_\alpha a_\alpha * D^{[\alpha]} u + A * u + \lambda u = f(x),\ x \in \mathbb{R}^n, \tag{10}$$

where l is a natural number, $a_\alpha = a_\alpha(x)$ are complex-valued functions, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, α_k are nonnegative integers, $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$, $\varepsilon_\alpha = \prod_{k=1}^n \varepsilon_k^{\frac{\alpha_k}{l}}$, ε_k are positive, λ is a complex parameter and $A = A(x)$ is a linear operator in a Banach space E for $x \in \mathbb{R}^n$ and

$$D^{[\alpha]} = D_{x_1}^{[\alpha_1]} D_{x_2}^{[\alpha_2]} ... D_{x_n}^{[\alpha_n]},\ D_{x_i}^{[\alpha_i]} = \left(\gamma_k(x_k) \frac{\partial}{\partial x_k} \right)^{\alpha_k},\ k = 1, 2, ..., n,$$

here $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$, $\gamma_k = \gamma_k(x_k)$ are positive measurable function on $\mathbb{R}$ such that $\gamma_k(x_k) \to 0$, when $x_k \to 0$ and

$$\lim_{|x_k| \to \infty} \gamma_k(x_k) = \infty.$$

Condition 3.1. *Let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$, $\gamma_k = \gamma_k(x_k)$ are positive measurable function on $\mathbb{R}$ such that $\gamma_k(x_k) \to 0$, when $x_k \to 0$. Moreover, assume that*

$$\int_0^{|a_k|} \gamma_k^{-1}(\tau) d\tau < \infty, \text{ for } a_k < \infty,\ k = 1, 2, ..., n$$

and

$$\lim_{|x_k|\to\infty}\gamma_k\left(x_k\right)=\infty.$$

Remark 3.1. *Let we put $\gamma_k\left(x_k\right)=\left|x_k\right|^{\sigma_k}$ for $0<\sigma_k<1$, $k=1,2,...,n$. It is not hard to see that these functions are satisfied the Condition 3.1.*

Remark 3.2. *Let the Condition 3.1 holds. Consider the following substitution*

$$y_k=\int_0^{x_k}\gamma_k^{-1}(z)dz,\ k=1,2,...,n. \tag{11}$$

It is clear that, under the substitution (11), $D^{[\alpha]}u$ transforms to $D^{\alpha}u$. Moreover, the spaces $L_p\left(\mathbb{R}^n;E\right), W_{p,\gamma}^{[l]}\left(\mathbb{R}^n;E\left(A\right),E\right)$ are mapped isomorphically onto the weighted spaces $L_{p,\tilde{\gamma}}(\mathbb{R}^n;E)$ and $W_{p,\tilde{\gamma}}^{l}(\mathbb{R}^n;E(A),E)$ respectively where,

$$\tilde{\gamma}=\widetilde{\gamma}(y)=\prod_{k=1}^{n}\widetilde{\gamma_k}(y),\ \widetilde{\gamma_k}(y)=\gamma_k(x(y)).$$

Moreover, under (11) the degenerate problem (10) considered in $L_p\left(\mathbb{R}^n;E\right)$ is transformed into the non degenerate problem (3) by replacing $A\left(x\right)$, $u\left(x\right)$, $f\left(x\right)$ with $\tilde{A}(y)$, $\tilde{u}(y)$, $\widetilde{f}(y)$ considered in $L_{p,\tilde{\gamma}}(\mathbb{R}^n;E)$, respectively, where

$$a_\alpha=a_\alpha\left(y\right)=a_\alpha(\widetilde{\gamma_1}(y),\widetilde{\gamma_2}(y),...,\widetilde{\gamma_n}(y)),\ \tilde{u}(y)=u(\widetilde{\gamma_1}(y),\widetilde{\gamma_2}(y),...,\widetilde{\gamma_n}(y)),$$

$$\tilde{A}(y)=A(\widetilde{\gamma_1}(y),\widetilde{\gamma_2}(y),...,\widetilde{\gamma_n}(y),\ \widetilde{f}(y)=f(\widetilde{\gamma_1}(y),\widetilde{\gamma_2}(y),...,\widetilde{\gamma_n}(y).$$

Let

$$\tilde{X}=L_p\left(\mathbb{R}^n;E\right),\ \tilde{Y}=W_{p,\gamma}^{[l]}\left(\mathbb{R}^n;E\left(A\right),E\right),\ p\in\left(1,\infty\right).$$

In this section, we show the following result:

Theorem 3.1. *Assume that the Conditions 2.1 and 3.1 are hold for $a_\alpha=a_\alpha\left(y\right)$ and E is a Banach space satisfying the multiplier condition with respect to weighted function γ and $p\in(1,\infty)$. Let $\hat{A}$ be a uniformly $R-$sectorial operator in E with $\varphi\in\left[0,\ \pi\right)$, $\lambda\in S_{\varphi_2}$ and $0\le\varphi+\varphi_1+\varphi_2<\pi$ for $A=A\left(y\right)$. Then for all $f\in\tilde{X}$, there is a unique solution of the problem (10) and the following coercive uniform estimation holds:*

$$\sum_{|\alpha|\le l}\varepsilon_\alpha\left|\lambda\right|^{1-\frac{|\alpha|}{l}}\left\|a_\alpha * D^{[\alpha]}u\right\|_{\tilde{X}}+\left\|A*u\right\|_{\tilde{X}}+\left|\lambda\right|\left\|u\right\|_{\tilde{X}}\le C\left\|f\right\|_{\tilde{X}}. \tag{12}$$

Proof. By Remark 3.2, the degenerate problem (10) is transformed into the non degenerate problem (3) considered in the weighted space $L_{p,\gamma}(\mathbb{R}^n;E)$. Then in view of Theorem 2.1 we obtain the assertion. □

4. Infinite system of the degenerate integro-differential equations with parameters

Consider the following infinity system of degenerate integro-differential equations

$$\sum_{|\alpha|\leq l}\varepsilon_{\alpha}a_{\alpha}*D^{[\alpha]}u_{m}+\sum_{j=1}^{\infty}d_{j}*u_{j}\left(x\right)=f_{m}\left(x\right),\ x\in\mathbb{R}^{n},\ \ m=1,2,...\ , \tag{13}$$

where $\varepsilon=(\varepsilon_1,\varepsilon_2,...,\varepsilon_n)$, ε_k are positive parameters and

$$\alpha=\left(\alpha_1,\alpha_2,...,\alpha_n\right),\ a_{\alpha}=a_{\alpha}\left(x\right)\varepsilon_{\alpha}=\prod_{k=1}^{n}\varepsilon_k^{\frac{\alpha_k}{l}},u_j=u_j\left(x\right),$$

$$D^{[\alpha]}=D_{x_1}^{[\alpha_1]}D_{x_2}^{[\alpha_2]}...D_{x_n}^{[\alpha_n]},\ D_{x_k}^{[\alpha_k]}=\left(\gamma\left(x\right)\frac{\partial}{\partial x_k}\right)^{\alpha_k}.$$

Here

$$\gamma\left(x\right)=\prod_{k=1}^{n}\left|x_k\right|^{\gamma},\ 0\leq\gamma<\frac{p-1}{n}.$$

Condition 4.1. *Assume that there exist positive constants C_1 and C_2 such that for $\{d_j\left(x\right)\}_1^{\infty}\in l_q$ for all $x\in\mathbb{R}^n$ and some $x_0\in\mathbb{R}^n$,*

$$C_1\left|d_j\left(x_0\right)\right|\leq\left|d_j\left(x\right)\right|\leq C_2\left|d_j\left(x_0\right)\right|.$$

Suppose $\hat{a}_{\alpha}$, $\hat{d}_m\in C^{(n)}\left(\mathbb{R}^n\right)$ and there exist positive constants M_i, $i=1,\ 2$ such that

$$|\xi|^{|\beta|}\left|D^{\beta}\hat{a}_{\alpha}(\xi)\right|\leq M_1,\ |\xi|^{|\beta|}\left|D^{\beta}\hat{d}_m(\xi)\right|\leq M_2\left|\hat{d}_m(\xi),\right|$$

$$\xi\in\mathbb{R}^n\backslash\left\{0\right\},\ \beta_k\in\left\{0,1\right\},0\leq|\beta|\leq n.$$

Let

$$D(x)=\left\{d_m(x)\right\},\ d_m>0,\ u=\left\{u_m\right\},\ D*u=\left\{d_m*u_m\right\},\ l_q(D)=$$

$$\left\{u\in l_q,\ \ \|u\|_{l_q(D)}=\left(\sum_{m=1}^{\infty}\left|d_m\left(x\right)*u_m\right|^q\right)^{\frac{1}{q}}<\infty\right\},\ 1<q<\infty.$$

Let Q_{ε} denote the differential operator in $L_p\left(\mathbb{R}^n;l_q\right)$ generated by (13).

Let

$$X=L_p\left(\mathbb{R}^n;l_q\right),\ Y=W_{p,\gamma}^{[l]}(\mathbb{R}^n;l_q(D),l_q),\ B=B\left(X\right).$$

Applying Theorem 2.1 we have:

Theorem 4.1. *Suppose that Condition 4.1 is satisfied. Then:*

(a) For all $f\left(x\right)=\left\{f_m\left(x\right)\right\}_1^{\infty}\in L_p\left(\mathbb{R}^n;l_q\left(D\right)\right)$, for $\lambda\in S_{\varphi}$, $\varphi\in[0,\pi)$ problem (4.1) has a unique solution $u=\left\{u_m\left(x\right)\right\}_1^{\infty}$ that belongs to Y and the following coercive estimation holds

$$\sum_{|\alpha|\leq l}\varepsilon_{\alpha}\left|\lambda\right|^{1-\frac{|\alpha|}{l}}\left\|a_{\alpha}*D^{[\alpha]}u\right\|_X+\left\|D*u\right\|_X\leq C\left\|f\right\|_X;$$

(b) For $\lambda\in S_{\varphi}$, there exists a resolvent $\left(Q_{\varepsilon}+\lambda\right)^{-1}$ and

$$\sum_{|\alpha|\leq l}\varepsilon_{\alpha}\left|\lambda\right|^{1-\frac{|\alpha|}{l}}\left\|a_{\alpha}*\left[D^{[\alpha]}\left(Q_{\varepsilon}+\lambda\right)^{-1}\right]\right\|_B+$$

$$\left\| D * (Q_\varepsilon + \lambda)^{-1} \right\|_B + \left\| \lambda (Q_\varepsilon + \lambda)^{-1} \right\|_B \leq C.$$

Proof. In fact, let $E = l_q$ and $A = [d_m(x)\,\delta_{jm}]$, $m, j = 1, 2, ...\infty$. Then

$$\hat{A}(\xi) = \left[\hat{d}_m(\xi)\,\delta_{jm}\right],\ D^\beta \hat{A}(\xi) = \left[D^\beta \hat{d}_m(\xi)\,\delta_{jm}\right],\ m, j = 1, 2, ...\infty.$$

It is easy to see that $\hat{A}(\xi)$ is uniformly $R-$sectorial in l_q and the all conditions of Theorem 3.1 hold. Moreover, by [5] we get that the space l_q satisfies the multiplier condition with respect to power weighted function $\gamma(x) = |x|^\gamma$, $-\frac{1}{n} < \gamma < \frac{p-1}{n}$ and $p \in (1, \infty)$. Therefore, by virtue of Theorem 3.1 we obtain the assertion (a). The assertion (b) is obtained from the Result 2.3 and Remark 3.2. □

Remark 4.1. *There are a lot of sectorial operators in concrete Banach spaces. Therefore, putting in (1) concrete Banach spaces instead of E and concrete sectorial differential, pseudo differential operators, or finite, infinite matrices, etc. instead of A, by virtue of Theorem 2.1, we can obtain the maximal regularity properties of different class of nonlocal equations or their systems, respectively.*

5. Conclusion

In this paper, we studied the maximal regularity properties of the linear nonlocal differential operator equations with parameters (1) in weighted $L_{p,\gamma}$ spaces. Moreover, we consider the degenerate nonlocal equation (10) and prove that a coercive uniform estimation holds. Using this result, we establish a coercive estimation for an infinite system of degenerate integro-differential equations with parameters. Note that by substituting specific Banach spaces for E and concrete differential operators, or finite, infinite matrices in place of A, in (1), we can derive the maximal regularity properties of different classes of nonlocal equations or their systems.

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