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Qualitative properties of fractional convolution elliptic and parabolic operators in Besov spaces

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Abstract

The maximal $B_{p,q}^{s}$-regularity properties of a fractional convolution elliptic equation is studied. Particularly, it is proven that the operator generated by this nonlocal elliptic equation is sectorial in $B_{p,q}^{s}$ and also is a generator of an analytic semigroup. Moreover, well-posedeness of nonlocal fractional parabolic equation in Besov spaces is obtained. Then by using the $B_{p,q}^{s}$-regularity properties of linear problem, the existence, uniqueness of maximal regular solution of corresponding fractional nonlinear equation is established.

Keywords Fractional-differential equations · Sobolev spaces · Elliptic equations · Maximal $B_{p,q}^{s}$-regularity · Parabolic equations · $B_{p,q}^{s}$-multipliers

Mathematics Subject Classification 47GXX · 35JXX · 47FXX · 47DXX · 43AXX

1 Introduction, notations and background

In the last years, fractional elliptic and parabolic equations have found many applications in physics (see [3, 4, 6, 8–12, 19] and the references therein). The regularity properties of fractional differential equations (FDEs) have been studied e.g. in [2, 3, 10, 13–16, 19–22]. Nonlocal fractional differential equations were studied, e.g. in [2,

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14, 15, 17, 19]. The main objective of the present paper is to discuss the $B_{p,q}^{s}\left(\mathbb{R}^{n}\right)$-maximal regularity of the nonlocal elliptic FDE with parameter

$$\sum_{|\alpha|\leq l} a_{\alpha} * D^{\alpha}u + \lambda u = f\left(x\right), x \in \mathbb{R}^{n}, \tag{1.1}$$

where a_{α} are complex valued functions, λ is a complex parameter, the convolution of functions $u = u\left(x\right)$ and $\upsilon = \upsilon\left(x\right)$ is defined by

$$u * \upsilon = \int_{\mathbb{R}^{n}} u\left(x - y\right)\upsilon\left(y\right)dy.$$

Here, l is a positive number, $\alpha = \left(\alpha_1, \alpha_2, ..., \alpha_n\right)$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, α_k are positive numbers. Moreover, the partial Riemann-Liouville fractional integral of order $\alpha_k > 0$ with respect to the k-th variable x_k is defined by

$$\left(I_{x_k}^{\alpha_k} f\right)\left(x\right) = \left(I_{x_k+}^{\alpha_k} f\right)\left(x\right) = \frac{1}{\Gamma\left(\alpha_k\right)}\int_{a_k}^{x_k} \frac{f\left(x_1, x_2, ...x_{k-1}, \tau, x_{k+1}, ...x_n\right)}{\left(x_k - \tau_k\right)^{1-\alpha_k}} d\tau_k.$$

The mixed Riemann-Liouville fractional integral of order $\alpha = \left(\alpha_1, \alpha_2, ..., \alpha_n\right)$ is defined by

$$\left(I_{x}^{\alpha} f\right)\left(x\right) = \left(I_{x+}^{\alpha} f\right)\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)}\int_{a_1}^{x_1} ... \int_{a_n}^{x_n} \frac{f\left(\tau\right)}{\left(x - \tau\right)^{1-\alpha}} d\tau,$$

where

$$x = \left(x_1, ..., x_n\right), \tau = \left(\tau_1, ..., \tau_n\right), \ \left(x - \tau\right)^{1-\alpha} = \prod_{k=1}^{n}\left(x_k - \tau_k\right)^{1-\alpha_k},$$

$$\Gamma\left(\alpha\right) = \prod_{k=1}^{n}\Gamma\left(\alpha_k\right).$$

Riemann-Liouville partial fractional derivative of order $\alpha_k > 0$ is defined by

$$\left(D_{k}^{\alpha_k} f\right)\left(x\right) = \left(-\frac{\partial}{\partial x_k}\right)^{l_k}\left(I_{x_k}^{l_k - \alpha_k} f\right)\left(x\right) \text{ for } l_k > \alpha_k + 1$$

and the corresponding mixed fractional derivative is defined by

$$D^{\alpha} = D_{1}^{\alpha_1} D_{2}^{\alpha_2} ... D_{n}^{\alpha_n},$$

where $\Gamma\left(\alpha_k\right)$ is the Gamma function for $\alpha_k > 0$ (see [6]§ 2.9 or [17]).

The maximal regularity properties of the linear problem (1.1) can be used to obtain the existence and uniqueness of solution to following fractional nonlinear equation

$$\sum_{|\alpha|\leq l} a_\alpha * D^\alpha u = \Phi\left(x, D^\sigma u\right) + f\left(x\right), x \in \mathbb{R}^n, \tag{1.3}$$

where, Φ is a nonlinear operator, $a_\alpha = a_\alpha\left(x\right)$ are complex-valued functions, f is a given function and D^σ denote all fractional differential operators such that $|\sigma| \leq l-1$.

In the mathematical modeling of physical phenomena, fractional differential equations are found to be better tools than their corresponding integer-order counterparts, for example, the description of anomalous diffusion via such equations leads to more informative and interesting model [8]. It is due to the nonlocal nature of fractional-order operators which can take into account the hereditary characteristics of the phenomena and processes involved in the modeling of real world problems.

Unlike the studies we recalled above, in this article (for the first time in our opinion), we show that the problem (1.1) is separable in Besov space $B^s_{p,q}\left(\mathbb{R}^n\right)$, i.e. for all $f \in B^s_{p,q}\left(\mathbb{R}^n\right)$ there exists a unique maximal regular solution such that all terms $a_\alpha * D^\alpha u$ in (1.1) also belong to $B^s_{p,q}\left(\mathbb{R}^n\right)$ and the resolvent of this differential operator under consideration develops with minimal growth.

Let E be a Banach space and $\mathbb{C}$ denote the set of complex numbers. Here, $L_p\left(\Omega; E\right)$ denotes the space of strong measurable E-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm given by

$$\|f\|_{L_p(\Omega;E)} = \left(\int_\Omega \|f\left(x\right)\|^p_E\, dx\right)^{\frac{1}{p}}, 1 \leq p < \infty,\ \|f\|_{L_\infty(\Omega;E)} = ess \sup_{x\in\Omega} \|f\left(x\right)\|_E .$$

Let $S = S\left(\mathbb{R}^n; E\right)$ denote the E-valued Schwartz space of rapidly decreasing, smooth functions and $S' = S'\left(\mathbb{R}^n; E\right)$ be a space of linear continued mapping from $S\left(\mathbb{R}^n\right)$ into E. $l_q\left(E\right)$ -denotes the E-valued sequence space of vectors $u = \{u_k\}^\infty_{k=0}$ with norm (see e.g [18, § 1.18]) given by

$$\|u\|_{l_q(E)} = \left[\sum_{k=0}^{\infty} \|u_k\|^q_E\right]^{\frac{1}{q}}, 1 \leq q < \infty,\ \|u\|_{l_\infty(E)} = \sup_k \|u_k\|_E .$$

Let F denote the Fourier transform defined by

$$\hat{u}\left(\xi\right) = F\left(u\right)\left(\xi\right) = \left(2\pi\right)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} u\left(x\right) dx \text{ for } u \in S(\mathbb{R}^n) \text{ and } x, \xi \in \mathbb{R}^n,$$

$$\check{u}\left(\xi\right) = F^{-1}\left(u\right)\left(\xi\right) = \left(2\pi\right)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\xi} u\left(x\right) dx, u \in S(\mathbb{R}^n), x, \xi \in \mathbb{R}^n.$$

Definition 1.1 A Banach space E has the Fourier type $r \in [1,2]$ provided the Fourier transform $\mathbb{F}$ defines a bounded linear operator from $L_r(\mathbb{R}^n; E)$ to $L_{r'}(\mathbb{R}^n; E)$ for $\frac{1}{r} + \frac{1}{r'} = 1$ (see e.g. [7], Remark 2.3).

In order to define abstract Besov spaces we consider the dyadic-like subsets $\{J_k\}_{k=0}^{\infty}$, $\{I_k\}_{k=0}^{\infty}$ of $\mathbb{R}^n$ and partition of unity $\{\varphi_k\}_{k=0}^{\infty}$ defined e.g. in ([18], § 2.3.1). Note the following useful properties are satisfied:

supp $\varphi_k \subset \bar{I}_k$ for each $k \in \mathbb{N}_0$; $\sum\limits_{k=0}^{\infty} \varphi_k(s) = 1$ for each $s \in \mathbb{R}^n$; $I_m \cap$ supp $\varphi_k = \oslash$ if $|m-k| > 1$; $\varphi_{k-1}(s) + \varphi_k(s) + \varphi_{k+1}(s) = 1$ for each $s \in$ supp φ_k and $k \in \mathbb{N}_0$.

Among the many equivalent descriptions of Besov spaces, the most useful one is given in terms of the so called Littlewood-Paley decomposition. This means that we consider $f \in S'$ as a distributional sum $f = \sum\limits_k f_k$ analytic functions f_k whose Fourier transforms have support in dyadic-like I_k and define the Besov norm in terms of the f_k's. The Besov space $Y^s = B^s_{p,q}(\mathbb{R}^n; E)$ is the space of all $f \in S'(\mathbb{R}^n; E)$ for which

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n;E)} = \left\| \left\{ 2^{ks} \left(\check{\varphi}_k * f \right) \right\}_{k=0}^{\infty} \right\|_{l_q(L_p(\mathbb{R}^n;E))} \tag{1.3}$$

$$= \begin{cases} \left[\sum\limits_{k=0}^{\infty} 2^{ksr} \left\| \check{\varphi}_k * f \right\|^r_{L_p(\mathbb{R}^n;E)} \right]^{\frac{1}{q}} < \infty, \text{ if } 1 \le r < \infty, \\ \sup\limits_{k \in \mathbb{N}_0} \left[\sum\limits_{k=0}^{\infty} 2^{ks} \left\| \check{\varphi}_k * f \right\|_{L_p(\mathbb{R}^n;E)} \right] < \infty, \text{ if } r = \infty. \end{cases}$$

$B^s_{p,q}(\mathbb{R}^n; E)$ with the norm in (1.3), is a Banach space (see e.g [[18], § 2.3.2] for $E = \mathbb{C}$). By reasoning as in [[18], § 2.3.2] for $E = \mathbb{C}$ it can be shown that different choices of $\{\varphi_k\}$ lead to equivalent norms on $B^s_{p,q}(\mathbb{R}^n; E)$.

Let Ω be a domain in $\mathbb{R}^n$. $B^s_{p,q}(\Omega; E)$ denotes the space of restrictions to Ω of all functions in $B^s = B^s_{p,q}(\mathbb{R}^n; E)$ with the norm given by

$$\|u\|_{B^s_{p,q}(\Omega;E)} = \inf_{g \in B^s, g|_{\Omega} = u} \|g\|_{B^s_{p,q}(\mathbb{R}^n;E)}.$$

Let $l \in (0,\infty)$, $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Here, $B^{l,s}_{p,q}(\Omega; E)$ denotes a Sobolev-Besov space of functions $u \in B^s_{p,q}(\Omega; E)$ that have fractional derivatives $D^l_k u \in B^s_{p,q}(\Omega; E)$ with the norm

$$\|u\|_{B^{l,s}_{p,q}(\Omega;E)} = \|u\|_{B^s_{p,q}(\Omega;E)} + \sum_{k=1}^{n} \left\| D^l_k u \right\|_{B^s_{p,q}(\Omega;E)} < \infty.$$

For $E = \mathbb{C}$ the abstract functional spaces $L_p(\Omega; E)$, $l_q(E)$, $B^s_{p,q}(\Omega; E)$, $B^{l,s}_{p,q}(\Omega; E)$ will be denoted by $L_p(\Omega)$, l_q, $B^s_{p,q}(\Omega)$, $B^{l,s}_{p,q}(\Omega)$, respectively.

A function $\Psi \in C\left(\mathbb{R}^n\right)$ is called a Fourier multiplier from $B_{p,q}^s\left(\mathbb{R}^n\right)$ to $B_{p,q}^s\left(\mathbb{R}^n\right)$ if the map

$$u \to \Lambda u = F^{-1}\Psi\left(\xi\right)Fu, u \in S\left(\mathbb{R}^n\right)$$

is well defined and extends to a bounded linear operator

$$\Lambda :\ B_{p,q}^s\left(\mathbb{R}^n\right) \to\ B_{p,q}^s\left(\mathbb{R}^n\right).$$

The function $u \in B_{p,q}^{l,s}\left(\mathbb{R}^n\right)$ satisfying the equation (1.1) a.e. on $\mathbb{R}^n$ is called a solution of (1.1).

We prove that the problem (1.1) has a unique solution $u \in B_{p,q}^{l,s}\left(\mathbb{R}^n\right)$ for $f \in B_{p,q}^s\left(\mathbb{R}^n\right)$ and the following uniform coercive estimate holds

$$\sum_{|\alpha|\leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\|a_\alpha * D^\alpha u\right\|_{B_{p,q}^s(\mathbb{R}^n)} \leq C\left\|f\right\|_{B_{p,q}^s(\mathbb{R}^n)}, \tag{1.4}$$

where the constant C is independent of λ. Let O be a linear operator generated by problem (1.1) for $\lambda = 0$, i.e.

$$D\left(O\right) = B_{p,q}^{l,s}\left(\mathbb{R}^n\right), Ou = \sum_{|\alpha|\leq l} a_\alpha * D^\alpha u \text{ for } u \in B_{p,q}^{l,s}\left(\mathbb{R}^n\right).$$

The estimate (1.4) implies that the operator O has a bounded inverse from $B_{p,q}^s\left(\mathbb{R}^n\right)$ into the space $B_{p,q}^{l,s}\left(\mathbb{R}^n\right)$. Particularly, from the estimate (1.4) we obtain that O is uniformly sectorial operator in $B_{p,q}^s\left(\mathbb{R}^n\right)$. By using the coercive properties of elliptic operator, we prove the well posedness of the Cauchy problem for the nonlocal fractional parabolic equation:

$$\frac{\partial u}{\partial t} + \sum_{|\alpha|\leq l} a_\alpha * D^\alpha u = f\left(t, x\right), u(0, x) = 0, \tag{1.5}$$

in the Besov space

$$Y^s = B_{p_1,q}^s\left(\mathbb{R}_+; B_{p,q}^s\left(\mathbb{R}^n\right)\right) \text{ for } p_1, p,\ q \in [1, \infty].$$

The function $u \in Y^s$ satisfying the equation (1.5) a.e. on $\mathbb{R}^n$ is called a solution of (1.5).

We show that problem (1.5) has a unique solution $u \in Y^s$ for each $f \in Y^s$ and satisfies the coercive estimate

$$\left\|\frac{\partial u}{\partial t}\right\|_{Y^s} + \sum_{|\alpha|\leq l} \left\|a_\alpha * D_x^\alpha u\right\|_{Y^s} + \left\|A * u\right\|_{Y^s} \leq M\left\|f\right\|_{Y^s}. \tag{1.6}$$

Let

$$S_\varphi = \left\{\lambda :\ \lambda \in \mathbb{C},\ |\arg \lambda| \leq \varphi\right\} \cup \left\{0\right\}, 0 \leq \varphi < \pi.$$

Here, $L(E_1, E_2)$ denotes the space of bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ it is denoted by $L(E)$. Let $D(A)$, $R(A)$ denote the domain and range of the linear operator in E, respectively. Let Ker A denote a null space of A. A closed linear operator A is said to be φ-sectorial (or sectorial for $\varphi = 0$) in a Banach space E with bound $M > 0$ if Ker $A = \{0\}$, $D(A)$ and $R(A)$ are dense on E, and $\left\|(A+\lambda I)^{-1}\right\|_{L(E)} \leq M|\lambda|^{-1}$ for all $\lambda \in S_\varphi$, $\varphi \in [0, \ \pi)$, where I is an identity operator in E. Sometimes $A+\lambda I$ will be written as $A+\lambda$ and will be denoted by A_λ. It is known [[18], §1.15.1] that the powers A^θ, $\theta \in (-\infty, \infty)$ for a sectorial operator A exist.

For any $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha_i \in [0, \infty)$, $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$ the function $(i\xi)^\alpha$ will be defined as:

$$(i\xi)^\alpha = \begin{cases} (i\xi_1)^{\alpha_1} ... (i\xi_n)^{\alpha_n}, \ \xi_1\xi_2...\xi_n \neq 0 \\ 0, \ \xi_1, \xi_2...\xi_n = 0, \end{cases} \tag{1.7}$$

where

$$(i\xi_k)^{\alpha_k} = \exp\left[\alpha_k\left(\ln|\xi_k| + i\frac{\pi}{2}\operatorname{sgn}\xi_k\right)\right], k = 1, 2, ..., n.$$

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say μ, we write C_μ. Moreover, for $u, \upsilon > 0$ the relations $u \lesssim \upsilon$, $u \approx \upsilon$ means that there exist positive constants C, C_1, C_2 independent on u and υ such that, respectively

$$u \leq C\upsilon, C_1\upsilon \leq u \leq C_2\upsilon.$$

The embedding theorems in vector valued spaces play a key role in the theory of PDEs. From [16] we obtain the estimates of lower order derivatives in $B^{s,l}_{p,q}(\mathbb{R}^n)$:

Theorem A_1 *Suppose* $1 \leq p \leq p_1 \leq \infty$, $q \in [1, \infty]$, l *is a positive integer and* $s \in (0, \infty)$ *with* $\varkappa = \frac{1}{l}\left[|\alpha| + n\left(\frac{1}{p} - \frac{1}{p_1}\right)\right] \leq 1$ *for* $0 \leq \mu \leq 1 - \varkappa$, *then the embedding*

$$D^\alpha B^{s,l}_{p,q}\left(\mathbb{R}^n\right) \subset B^s_{p_1,q}\left(\mathbb{R}^n\right)$$

is continuous and there exists a constant $C_\mu > 0$, *depending only on* μ *such that*

$$\left\|D^\alpha u\right\|_{B^s_{p_1,q}(\mathbb{R}^n)} \leq C_\mu\left[h^\mu \|u\|_{B^{s,l}_{p,q}(\mathbb{R}^n)} + h^{-(1-\mu)}\|u\|_{B^s_{p,q}(\mathbb{R}^n)}\right]$$

for all $u \in B^{s,l}_{p,q}(\mathbb{R}^n)$ *and* $0 < h \leq h_0 < \infty$.

2 Nonlocal fractional elliptic equation

Consider the problem (1.1). Let $[\eta]$ denote the integer part of the real number η.

Condition 2.1 Assume $a_\alpha \in L_1(\mathbb{R}^n)$ such that

$$L(\xi) \neq 0, L(\xi) = \sum_{|\alpha| \leq l} \hat{a}_\alpha(\xi)(i\xi)^\alpha \in S_{\varphi_1} \text{ for all } \xi \in \mathbb{R}^n \tag{2.1}$$

and

$$|L(\xi)| \geq C \sum_{k=1}^{n} \left|\hat{a}_{\alpha(l,k)}\right| |\xi_k|^l$$

for

$$\alpha(l,k) = (0, 0, ..., l, 0, 0, ..., 0), \text{ i.e. } \alpha_i = 0, i \neq k,$$

where $(i\xi)^\alpha$ is defined by (1.7).

Moreover, suppose that $\hat{a}_\alpha \in C^{(n)}(\mathbb{R}^n)$ and

$$(1 + |\xi|)^{|\beta|} \left|D_\xi^\beta \hat{a}_\alpha(\xi)\right| \leq C_1, \beta_k \in \{0, 1\} \tag{2.2}$$

for

$$k = 1, 2, ...n, \beta = (\beta_1, \beta_2, ..., \beta_n), \xi \in \mathbb{R}^n, 0 \leq |\beta| \leq \left[\frac{n}{2}\right] + 1.$$

Remark 2.1 Here, the conditions on the functions $a_\alpha(x)$ are given in terms of their Fourier transform $\hat{a}_\alpha(\xi)$. Appropriate conditions can be made on the function a_α. For example let $a_\alpha \in S(\mathbb{R}^n)$. Since Fourier transform F transforms $S(\mathbb{R}^n)$ into $S(\mathbb{R}^n)$, then $\hat{a}_\alpha(\xi) = (Fa_\alpha)(\xi) \in S(\mathbb{R}^n)$. Hence, $\hat{a}_\alpha(\xi)$ satisfies the assumption (2.2).

Consider operator functions

$$\sigma_1(\xi, \lambda) = \sum_{|\alpha| \leq l} |\lambda|^{1 - \frac{|\alpha|}{l}} \hat{a}_\alpha(\xi)(i\xi)^\alpha \sigma_0(\xi, \lambda), \tag{2.3}$$

where

$$\sigma_0(\xi, \lambda) = [L(\xi) + \lambda]^{-1}.$$

Let

$$X = B_{p,q}^s(\mathbb{R}^n), Y = B_{p,q}^{l,s}(\mathbb{R}^n).$$

In this section we prove the following:

Theorem 2.1 *Assume that Condition 2.1 is satisfied,* $p, q \in [1, \infty]$ *and* $\lambda \in S_{\varphi_2}$. *Then for* $f \in X$, $0 \leq \varphi_1 < \pi - \varphi_2$ *and* $\varphi_1 + \varphi_2 \leq \varphi$ *there is a unique solution* u *of the equation* (1.1) *belonging to* Y *and the following coercive uniform estimate holds*

$$\sum_{|\alpha| \leq l} |\lambda|^{1 - \frac{|\alpha|}{l}} \left\|a * D^\alpha u\right\|_X + \|u\|_X \leq C\|f\|_X. \tag{2.4}$$

To prove Theorem 2.1, we need the following lemmas:

Lemma 2.1 *Assume that Condition 2.1 holds and* $\lambda \in S_{\varphi_2}$*with* $\varphi_2 \in [0, \ \pi)$, *where* $\varphi_1 + \varphi_2 < \pi$, *then the operator functions* $\sigma_i \ (\xi, \lambda)$ *are bounded uniformly with respect to* $\xi \in \mathbb{R}^n$ *and* $\lambda \in S_{\varphi_2}$, *i.e.,*

$$\sup_{\xi \in \mathbb{R}^n, \lambda \in S_{\varphi_2}} |\sigma_i \ (\xi, \lambda)| \leq C, \, i = 0, 1.$$

Proof By virtue of [[5], Lemma 2.3], for $L(\xi) \in S_{\varphi_1}$, $\lambda \in S_{\varphi_2} \cap \{0\}$ and $\varphi_1 + \varphi_2 < \pi$ there exists a positive constant C such that

$$|\lambda + L\ (\xi)| \geq C\ (|\lambda| + |L\ (\xi)|)\,. \tag{2.5}$$

Since $L(\xi) \in S_{\varphi_1}$, in view of Condition 2.1 and (2.5) the function $\sigma_0\ (\xi, \lambda)$ is uniformly bounded for all $\xi \in \mathbb{R}^n$, $\lambda \in S_{\varphi_2}$, i.e.

$$|\sigma_0\ (\xi, \lambda)| \leq (|\lambda| + |L\ (\xi)|)^{-1} \leq M_0.$$

Next, let us consider σ_1. It is clear to see that

$$|\sigma_1\ (\xi, \lambda)| \leq C\,|\lambda| \sum_{|\alpha| \leq l} \prod_{k=1}^{n} \left[|\xi|\,|\lambda|^{-\frac{1}{l}}\right]^{\alpha_k} |\sigma_0\ (\xi, \lambda)|\,. \tag{2.6}$$

By setting $y_k = \left(|\lambda|^{-\frac{1}{l}}\ |\xi_k|\right)^{\alpha_k}$ in the following well known inequality

$$y_1^{\alpha_1} y_2^{\alpha_2} ... y_n^{\alpha_n} \leq C \left(1 + \sum_{k=1}^{n} y_k^l\right), \, y_k \geq 0, \ |\alpha| \leq l \tag{2.7}$$

we get

$$\|\sigma_1\ (\xi, \lambda)\|_{B(E)} \leq C \sum_{|\alpha| \leq l} |\lambda| \left[1 + \sum_{k=1}^{n} |\xi_k|^l\ |\lambda|^{-1}\right] |\lambda + L\ (\xi)|^{-1}\,.$$

Taking into account Condition 2.1 and (2.6) − (2.7) we obtain

$$|\sigma_1\ (\xi, \lambda)| \leq C \left(|\lambda| + \sum_{k=1}^{n} |\xi_k|^l\right) (|\lambda| + |L\ (\xi)|)^{-1} \leq C.$$

□

Lemma 2.2 *Assume that Condition 2.1 holds and* $\lambda \in S_{\varphi_2}$ *with* $\varphi_2 < \pi - \varphi_1$. *Then, operators* $(1 + |\xi|)^{|\beta|}\ D_\xi^\beta \sigma_i\ (\xi, \lambda)$ *are uniformly bounded, i.e.*

$$\sup_{\xi \in \mathbb{R}^n, \lambda \in S_{\varphi_2}} (1 + |\xi|)^{|\beta|}\ D_\xi^\beta \sigma_i\ (\xi, \lambda) \leq C, \, i = 0, 1.$$

Proof Consider the term $|\xi|^{|\beta|}\, D_{\xi}^{\beta}\sigma_0\left(\xi,\lambda\right)$. By using Condition 2.1, by virtue of [[5], Lemma 2.3] and by reasoning as in Lemma 2.1, we get

$$|\xi_k|\left|D_{\xi_k}\sigma_0\left(\xi,\lambda\right)\right| \tag{2.8}$$

$$\lesssim \left(|L\left(\xi\right)|+|\lambda|\right)^{-2}\sum_{|\alpha|\leq l}|\xi_k|\left|D_{\xi_k}\hat{a}_{\alpha}(\xi)\right|\prod_{k=1}^{n}|\xi_k|^{\alpha_k}+\left|\hat{a}_{\alpha}(\xi)\right|\prod_{k=1}^{n}|\xi_k|^{\alpha_k}.$$

Then by the nature of symbol $L\left(\xi\right)$ defined by (2.1), by using (1.7), by assumption on $\hat{a}_{\alpha}(\xi)$, and by (2.1) we get the uniform estimate

$$|\xi_k|\left|D_{\xi_k}\sigma_0\left(\xi,\lambda\right)\right| \tag{2.9}$$

$$\lesssim \left[\sum_{k=1}^{n}\left|\hat{a}_{\alpha(l,k)}\right||\xi_k|^{l}+|\lambda|\right]^{-2}\sum_{|\alpha|\leq l}\left(\sum_{k=1}^{n}|\xi_k|^{l_k}\right)^{-2}\leq C.$$

The same time, by similar way, we get

$$\left|D_{\xi_k}\sigma_0\left(\xi,\lambda\right)\right| \tag{2.10}$$

$$\lesssim \left(|L\left(\xi\right)|+|\lambda|\right)^{-2}\sum_{|\alpha|\leq l}|\xi_k|\left|D_{\xi_k}\hat{a}_{\alpha}(\xi)\right|\prod_{k=1}^{n}|\xi_k|^{\alpha_k}+\left|\hat{a}_{\alpha}(\xi)\right|\prod_{k=1}^{n}|\xi_k|^{-1}|\xi_k|^{\alpha_k}.$$

By estimate (2.2) we have the following uniform estimates

$$\left|\hat{a}_{\alpha}(\xi)\right||\xi_k|^{-1}\leq C. \tag{2.11}$$

Hence, from (2.10) and (2.11) the uniform estimates are deduced

$$\left|D_{\xi_k}\sigma_0\left(\xi,\lambda\right)\right|\leq C \tag{2.12}$$

In view of (2.9) and (2.12) we have the following uniform estimates

$$\sup_{\xi\in\mathbb{R}^n,\lambda\in S_{\varphi_2}}\left(1+|\xi_k|\right)D_{\xi_k}\sigma_0\left(\xi,\lambda\right)\leq C,\ k=1,2,...,n. \tag{2.13}$$

It easy to see that the operators $\left(1+|\xi|\right)^{\beta}\,D^{\beta}\sigma_0\left(\xi,\lambda\right)$ contain the similar terms as in $\left(1+|\xi_k|\right)\left|D_{\xi_k}\sigma_0\left(\xi,\lambda\right)\right|$ for all $\beta_k\in\{0,1\}$. Hence, from (2.13) we get

$$\sup_{\xi\in\mathbb{R}^n,\lambda\in S_{\varphi_2}}\left(1+|\xi|\right)^{|\beta|}\left|D_{\xi}^{\beta}\sigma_0\left(\xi,\lambda\right)\right|<\infty. \tag{2.14}$$

In a similar way, by using Condition 2.1, the representations of function $\sigma_1(\xi,\lambda)$ in (2.3) by $\sigma_0(\xi,\lambda)$ and the estimates (2.5) – (2.7), we obtain the uniform estimates

$$\sup_{\xi\in\mathbb{R}^n,\lambda\in S_{\varphi_2}} (1+|\xi|)^{|\beta|}\left|D_\xi^\beta\sigma_1(\xi,\lambda)\right| < \infty. \tag{2.15}$$

Hence, from (2.14) and (2.15) we obtain the conclusion of Lemma 2.2. □

Remark 2.1 By definition of E-valued Besov spaces $B_{p,q}^s(\mathbb{R}^n;E)$ it is clear to see that

$$B_{p,q}^s\left(\mathbb{R}^n\right) = B_{p,q}^s\left(\mathbb{R}^n;\mathbb{C}\right),$$

where $\mathbb{C}$ denotes the set of all complex numbers. Since $\mathbb{C}$ is a Hilbert space then by virtue of [[7], Remark 2.2], $\mathbb{C}$ has Fourier type 2, i.e. $r=2$ (See Definition 1.1). Hence, in view of [[7], Corollary 4.11] in assumption (2.2) of Condition 2.1 we required $r=2$ to be Fourier multipliers of functions $\sigma_i(\xi,\lambda)$.

Here, we assume that $D_k^{\alpha_k}$ in (1.1) denote the Riesz fractional derivatives (see e.g. [6] § 2.10).

Proof (Theorem 2.1) By applying the Fourier transform to equation (1.1), we get

$$\hat{u}(\xi) = \sigma_0(\xi,\lambda)\hat{f}(\xi), \sigma_0(\xi,\lambda) = [L(\xi)+\lambda]^{-1}. \tag{2.16}$$

Hence, the solution of (1.1) can be represented as $u(x) = F^{-1}\left[\sigma_0(\xi,\lambda)\hat{f}(\xi)\right]$ and by Lemma 2.1 there are positive constants C_1 and C_2 such that

$$C_1|\lambda|\|u\|_X \le \left\|F^{-1}\left[\lambda\sigma_0(\xi,\lambda)\hat{f}\right]\right\|_X \le C_2\|u\|_X,$$

$$C_1\sum_{|\alpha|\le l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_\alpha * D^\alpha u\right\|_X \le \left\|F^{-1}\left[\sigma_2(\xi,\lambda)\hat{f}\right]\right\|_X 0 \tag{2.17}$$

$$\le C_2\sum_{|\alpha|\le l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_\alpha * D^\alpha u\right\|_X.$$

By (2.16) and (2.17) it is sufficient to show that the functions $\sigma_i(\xi,\lambda)$ are Fourier multipliers in X. But, by Lemma 2.2 and by virtue of Fourier multiplier theorem in Besov spaces $X = B_{p,q}^s(\mathbb{R}^n)$ (see e.g [[7], Corollary 4.11]) we get that $\sigma_i(\xi,\lambda)$ are Fourier multipliers in X, which concludes the proof. □

Result 2.1 Theorem 2.2 implies that the operator O is separable in X, i.e. for all $f\in X$ there is a unique solution $u\in Y$ of the problem (1.1), all terms of equation (1.1) are also from X and there are positive constants C_1 and C_2 so that

$$C_1\|Ou\|_X \le \sum_{|\alpha|\le l}\left\|a_\alpha * D^\alpha u\right\|_X + \|u\|_X \le C_2\|Ou\|_X.$$

Indeed, if we put $\lambda = 1$ in (2.3), by Theorem 2.1 we get the second inequality. So it remains to prove the first estimate. The first inequality is equivalent to the following estimate

$$\sum_{|\alpha|\leq l}\left\|F^{-1}\left[\hat{a}_{\alpha}\left(i\xi\right)^{\alpha}\hat{u}\right]\right\|_{X}\leq\sum_{|\alpha|\leq l}\left\|F^{-1}\left[\hat{a}_{\alpha}\left(i\xi\right)^{\alpha}\sigma_{0}\left(\xi,\lambda\right)\hat{f}\left(\xi\right)\right]\right\|_{X}.$$

So, it suffices to show that the operator functions

$$\sigma_{0}\left(\xi,\lambda\right),\ \sum_{|\alpha|\leq l}\hat{a}_{\alpha}\left(i\xi\right)^{\alpha}\sigma_{0}\left(\xi,\lambda\right)$$

are uniform Fourier multipliers in X. This fact is proved in a similar way as in the proof of Theorem 2.1.

From Theorem 2.1, we have:

Result 2.2 Assume that the all conditions of Theorem 2.1 are satisfied. Then, for all $\lambda\in S_{\varphi}$ the resolvent of operator O exists and the following sharp coercive uniform estimate holds

$$\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_{\alpha}*D^{\alpha}\left(O+\lambda\right)^{-1}\right\|_{L(X)}+\left\|\left(O+\lambda\right)^{-1}\right\|_{L(X)}\leq C. \tag{2.18}$$

Indeed, we infer from Theorem 2.1 that the operator $O+\lambda$ has a bounded inverse from X to Y. So, the solution u of the equation (1.1) can be expressed as $u\left(x\right)=\left(O+\lambda\right)^{-1}f$ for all $f\in X$. Then estimate (2.4) implies (2.18).

Remark 2.2 From (2.18), we deduced the following uniform estimate

$$\left\|\left(O+\lambda\right)^{-1}\right\|_{L(X)}\leq C.$$

Theorem 2.2 *Assume that Condition 2.1 is satisfied,* $p, q\in[1,\infty]$ *and* $\lambda\in S_{\varphi_2}$ *and*

$$\left|L\left(\xi\right)\right|\geq C\sum_{k=1}^{n}\left|\xi_{k}\right|^{l}.$$

Then for $f\in X$, $0\leq\varphi_1<\pi-\varphi_2$ *and* $\varphi_1+\varphi_2\leq\varphi$ *there is a unique solution* u *of* (1.1) *belonging to* Y *and the following coercive uniformly estimate holds*

$$\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|D^{\alpha}u\right\|_{X}+\left\|u\right\|_{X}\leq C\left\|f\right\|_{X}. \tag{2.19}$$

Proof The estimate (2.19) can be derived by the same reasoning as in Theorem 2.1. □

From Theorem 2.2, we have the following results:

Result 2.3 There are positive constants C_1 and C_2 so that

$$C_1 \left\| Ou \right\|_X \leq \sum_{|\alpha|\leq l} \left\| D^{\alpha} u \right\|_X + \left\| Au \right\|_X \leq C_2 \left\| Ou \right\|_X . \tag{2.20}$$

Result 2.4 Assume all conditions of Theorem 2.2 hold. Then, for all $\lambda \in S_{\varphi_2}$ the resolvent of operator O exists and the following sharp uniform estimate holds

$$\sum_{|\alpha|\leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\| D^{\alpha} \left(O+\lambda\right)^{-1} \right\|_{L(X)} + \left\| \left(O+\lambda\right)^{-1} \right\|_{L(X)} \leq C. \tag{2.21}$$

Result 2.5 Theorem 2.2 particularly implies that the operator O is sectorial in X. Then the operators O^{ν} are generators of analytic semigroups in X for $\nu \leq \frac{1}{2}$ (see e.g. [18], §1.14.5).

3 The Cauchy problem for fractional parabolic equation

In this section, we shall consider the following Cauchy problem for the parabolic convolution FDE

$$\frac{\partial u}{\partial t} + \sum_{|\alpha|\leq l} a_{\alpha} * D_x^{\alpha} u = f\left(t,x\right), u(0,x) = 0, t \in \mathbb{R}_+, x \in \mathbb{R}^n, \tag{3.1}$$

where a is a complex number, D_x^{α} is the fractional derivative in x defined by (1.2).

By applying Theorem 2.1 we establish the maximal regularity of the problem (3.1) in mixed Besov spaces. Let O denote the operator generated by problem (1.1) for $\lambda = 0$. Let

$$X = B_{p,q}^{s}\left(\mathbb{R}^n\right), Y_{\mathbf{p}}^{s} = B_{p_1,q}^{s}\left(\mathbb{R}_+; X\right), Y_0 = B_{\mathbf{p},q}^{s}\left(\mathbb{R}_+^{n+1}\right),$$
$$\mathbf{p} = \left(p_1, p\right).$$

Let $Y^{1,l,s} = B_{\mathbf{p},q}^{1,l,s}\left(\mathbb{R}_+^{n+1}\right)$ denote the space of all functions $u \in X$ possessing the derivative $\frac{\partial u}{\partial t} \in Y_0$ and fractional derivatives $D_x^{\alpha} u \in Y_0$ for $|\alpha| \leq l$ with the norm

$$\left\| u \right\|_{Y^{1,l,s}} = \left\| u \right\|_{Y_0} + \left\| \frac{\partial u}{\partial t} \right\|_{Y_0} + \sum_{|\alpha|\leq l} \left\| D_x^{\alpha} u \right\|_{Y_0},$$

where $u = u\left(t,x\right)$.

Now, we are ready to state the main result of this section.

Theorem 3.1 *Assume that the all conditions of Theorem 2.2 hold for* $\varphi \in \left(\frac{\pi}{2}, \pi\right)$ *and* $p_1, p, q \in [1, \infty]$. *Then for* $f \in Y_0$ *problem* (3.1) *has a unique solution* $u \in Y^{1,l,s}$

satisfying the following unform coercive estimate

$$\left\|\frac{\partial u}{\partial t}\right\|_{Y_0}+\sum_{|\alpha|\leq l}\left\|a_\alpha * D_x^\alpha u\right\|_{Y_0}+\|u\|_{Y_0}\leq C\,\|f\|_{Y_{\mathbf{p}}^s}.$$

Proof By definitions of X and the mixed space $B_{\mathbf{p},q}^s\left(\mathbb{R}_+^{n+1}\right)$ for $\mathbf{p}=(p,\,p_1)$, we have

$$\|u\|_{Y_{\mathbf{p}}^s}=\|u\|_{B_{p_1,q}^s(\mathbb{R}_+;X)}=\left\|\left\{2^{ks}\left(\check{\varphi}_k * f\right)\right\}_{k=0}^{\infty}\right\|_{l_q\left(L_{p_1}(\mathbb{R}^n;X)\right)} \tag{3.2}$$

$$=\left[\sum_{k=0}^{\infty}\left\|\left\{2^{ks}\left(\check{\varphi}_k * f\right)\right\}^q\right\|_{L_{p_1}(\mathbb{R}^n;X)}\right]^{\frac{1}{q}}$$

$$=\left[\sum_{k=0}^{\infty}\left(\int_{\mathbb{R}^n}\left\|\left\{2^{ks}\left(\check{\varphi}_k * f\right)\right\}^q\right\|_{B_{p,q}^s(\mathbb{R}^n)}^{p_1}dx\right)^{\frac{1}{p_1}}\right]^{\frac{1}{q}}\geq\|u\|_{B_{\mathbf{p},q}^s\left(\mathbb{R}_+^{n+1}\right)}.$$

So, we get the following embedding

$$Y_{\mathbf{p}}^s\subset B_{\mathbf{p},q}^s\left(\mathbb{R}_+^{n+1}\right).$$

Hence, the problem (3.1) can be expressed as the following Cauchy problem for the parabolic equation

$$\frac{du}{dt}+Ou\,(t)=f\,(t)\,,\ u\,(0)=0,\,t\in\mathbb{R}_+. \tag{3.3}$$

By assumption and by Result 2.5 the operator O generates an analytic semigroup in X_0. Then, by virtue of [1], § 8, we obtain that for $f\in Y$ problem (3.3) has a unique solution $u\in B_{p_1,q}^{1,s}\left(\mathbb{R}_+;\,D\,(O)\,,\,Y_{\mathbf{p}}^s\right)$ satisfying the following estimate

$$\left\|\frac{du}{dt}\right\|_{B_{p_1,q}^s(\mathbb{R}_+;X)}+\|Ou\|_{B_{p_1,q}^s(\mathbb{R}_+;X)}\leq C\,\|f\|_{B_{p_1,q}^s(\mathbb{R}_+;X)}\,.$$

□

Now, we can prove the following result:

Theorem 3.2 *Assume that the all conditions of Theorem 2.2 hold for* $\varphi\in\left(\frac{\pi}{2},\pi\right)$, $p_1,\ p,\ q\in[1,\infty]$. *Then for* $f\in Y$ *problem* (3.1) *has a unique solution* $u\in Y^{1,l,s}$

satisfying the following unform coercive estimate

$$\left\| \frac{\partial u}{\partial t} \right\|_{Y_0} + \sum_{|\alpha| \leq l} \left\| D_x^\alpha u \right\|_{Y_0} + \|u\|_{Y_0} \leq C \|f\|_{Y_{\mathbf{p}}^s}.$$

Proof Indeed, by reasoning as in Theorem 3.1 and using Theorem 2.2 instead of Theorem 2.1, we obtain the conclusion. □

4 The quasilinear fractional nonlinear elliptic equation

Consider nonlinear fractional nonlocal elliptic equation (1.3) . Let

$$\mathrm{X} = B_{p,q}^{s}\left(\mathbb{R}^n\right), Y = B_{p,q}^{l,s}\left(\mathbb{R}^n\right), E_0 = \prod_{|\sigma| \leq l_0 - 1} E_\sigma,$$

$$E_\sigma = B_{p,q}^{l-[\sigma]-\frac{n}{p},s}\left(\mathbb{R}^{n-1}\right), l_0 = \left[l - \frac{n}{p}\right].$$

Remark 4.1 By the trace theorem in Sobolev and Besov spaces (see e.g. [18], § 4.7) the embedding $D^\sigma Y \in E_\sigma$ is continuous and, we have

$$\prod_{|\sigma| \leq l_0 - 1} \left\| D^\sigma u \right\|_{C(\mathbb{R}^n; E_\sigma)} = \prod_{|\sigma| \leq l_0 - 1} \sup_{x \in \mathbb{R}^n} \left\| D^\sigma u(x) \right\|_{E_\sigma} \leq \|u\|_Y \tag{4.1}$$

for

$$u \in Y,\ U_\sigma = \{u_\sigma\},\ u_\sigma = D^\sigma u(\cdot),\ |\sigma| \leq l_0 - 1.$$

Let

$$O_r = \left\{ \upsilon \in E_0,\ \|\upsilon\|_{E_0} \leq r \right\},\ 0 < r \leq r_0.$$

Consider the linear fractional equation

$$\sum_{|\alpha| \leq l} a_\alpha * D^\alpha w = g(x). \tag{4.2}$$

From Theorem 2.1 we conclude that the problem (4.2) has a unique solution $w \in Y$ and the following coercive estimate holds

$$\sum_{|\alpha| \leq l} \left\| a_\alpha * D^\alpha w \right\|_X + \|w\|_X \leq M \|g\|_X \tag{4.3}$$

for all $g \in X$ and $p, q \in [1, \infty]$. Moreover, from the estimate (2.18) we deduced that the operator $(O + \lambda)^{-1}$ is bounded for $\lambda = 0$, i.e. the operator O has a bounded inverse O^{-1}, where O is the operator generated by problem (4.2).

Condition 4.1 Assume $\Phi : \mathbb{R}^n \times E_0 \to \mathbb{C}$ is a measurable function for each $U \in E_0$ and $\Phi(x, U) \in X$. Moreover, for each $r > 0$ there exist the positive functions $h_k(.) \in X$ such that for $U, \bar{U} \in O_r$ and

$$|\Phi(x, U)|_{\mathbb{C}} \leq h_1(x) \|U\|_{E_0}, \quad \left\|\Phi(x, U) - \Phi\left(x, \bar{U}\right)\right\|_{\mathbb{C}} \leq h_2(x) \left\|U - \bar{U}\right\|_{E_0},$$

where

$$\|h_k\|_X < M^{-1}, k = 1, 2.$$

Theorem 4.1 *Assume that the Conditions 2.1 and 4.1 are satisfied. Then, there exist an r with $0 < r \leq r_0$ and $\delta > 0$ such that for each $f \in X$ with $\|f\|_X \leq \delta$ there exists a unique solution $u \in Y$ of equation (1.3) with $\|u\|_Y \leq r$.*

Proof We want to solve problem (1.3) locally by means of maximal regularity of the linear problem (4.2) via the contraction mapping theorem. Consider the function

$$g(x) = \Phi(x, U_\sigma) + f(x), \tag{4.4}$$

where

$$U_\sigma = \left\{D^\sigma \upsilon\right\}, \ |\sigma| \leq l_0 - 1, \upsilon \in Y.$$

Consider the following ball

$$B_r = \{u \in Y, \ \|u\|_Y \leq r\}.$$

Let $f \in X$ such that $\|f\|_X \leq \delta$ and $\upsilon \in B_r$. Thus, by assumption

$$|g(x)| \leq h_1(x) \|U_\sigma\|_{E_\sigma} + |f(x)|$$

for a.e. $x \in \mathbb{R}^n$. By using (4.1), we have

$$\|U_\sigma\|_{E_0} < \|\upsilon\|_Y \leq r. \tag{4.5}$$

Moreover, choose $0 < r \leq r_0$ and δ such that

$$\delta + M \ \|h_1\|_X < \frac{1}{2}. \tag{4.6}$$

Then, from (4.5), (4.6) it follows that

$$M \|Q\|_X \leq r. \tag{4.7}$$

Define a map $Q\upsilon = u$ on B_r, where u is a solution of (1.3) for the inhomogeneity $g(x)$ defined by (4.4). We want to show that $Q(B_r) \subset B_r$ and that Q is a contraction operator in Y. By (4.7), Q maps the set B_r into itself. We show that Q is a strict contraction. In fact, let

$$u_1 = Q\upsilon_1, u_2 = Q\upsilon_2, \upsilon_1, \upsilon_2 \in B_r.$$

It is clear to see that $u_1 - u_2$ is a solution of the problem (4.2) for the inhomogeneity

$$g = \Phi(x, \upsilon_1) - \Phi(x, \upsilon_2).$$

As before by using the assumption on Φ, we have

$$\|Q\upsilon_1 - Q\upsilon_2\|_Y \leq M\|g\|_X \leq M\|h_2\|_X\|\upsilon_1 - \upsilon_2\|_Y.$$

Hence, by virtue of condition on h_2, Q is a strict contraction. By Banach's fixed point theorem there exists a unique fixed point $u \in Y$ of Q which implies the assertion. □

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Conflicts of interest The authors declare that they have no conflict of interest.

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