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Nonlocal fractional elliptic equations and applications

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Abstract

The L_p-coercive properties of a nonlocal fractional elliptic equation is studied. Particularly, it is proved that the fractional elliptic operator generated by this equation is sectorial in L_p space and also is a generator of an analytic semigroup. Moreover, by using the L_p-separability properties of the given elliptic operator the maximal regularity of the corresponding nonlocal fractional parabolic equation is established.

KEYWORDS

elliptic equations, fractional-differential equations, L_p-multipliers, maximal L_p regularity, parabolic equations, Sobolev spaces

1 | INTRODUCTION, NOTATIONS, AND BACKGROUND

In the last years, fractional elliptic and parabolic equations have found many applications in physics (see [1–4] and the references therein). The regularity properties of fractional differential equations (FDEs) have been studied, for example, in [2, 5–12]. The main objective of the present paper is to discuss the $L_p\left(\mathbb{R}^n\right)$ -maximal regularity of the nonlocal FDE with parameter

$$\sum_{|\alpha|\leq l} a_\alpha * D^\alpha u + \lambda u = f(x), x \in \mathbb{R}^n \tag{1.1}$$

where a_α are complex valued functions, λ is a complex parameter, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n}$ for $\alpha_i \in [0, \infty)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$. Here, D^{α_k} are Riemann–Liouville type fractional partial derivatives of order $\alpha_k \in (1, 2)$ with respect to x_k, that is,

$$D_k^{\alpha_k} u = \frac{1}{\Gamma(2-\alpha_k)} \frac{\partial^2}{\partial x_k^2} \int_0^{x_k} \frac{u(y)dy}{(x_k - y)^{\alpha_k - 1}}, k = 1, 2, \dots, n, 1.2 \tag{1.2}$$

$\Gamma(\gamma)$ is Gamma function for $\gamma > 0$ (see, e.g., [3, 13, 14]), $a_\alpha * D^\alpha u$ denotes a convolution of a_α and $D^\alpha u$.

For, we have

$$D_k^{\frac{1}{2}}u = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \frac{\partial^2}{\partial x_k^2} \int_0^{x_k} (x_k - y)^{\frac{1}{2}} u(y)dy, k = 1, 2, \ldots, n.$$

Here, $L_p(\Omega)$ denotes the space of strongly measurable complex-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm given by

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}, 1 \leq p < \infty .$$

Let $S\left(\mathbb{R}^n\right)$ (see, e.g., section 2.2.1 in [15]) denote the complex-valued Schwartz class, that is, the space of all rapidly decreasing smooth functions on $\mathbb{R}^n$ equipped with its usual topology generated by seminorms.

A function $\Psi \in C\left(\mathbb{R}^n\right)$ is called a Fourier multiplier from $L_p\left(\mathbb{R}^n\right)$ to $L_p\left(\mathbb{R}^n\right)$ (see, e.g., section 2.2.2 in [15] or [16]) if the map

$$u \to \Lambda u = F^{-1}\Psi(\xi)Fu, \ u \in S\left(\mathbb{R}^n\right)$$

is well defined and extends to a bounded linear operator

$$\Lambda : \ L_p\left(\mathbb{R}^n\right) \to \ L_p\left(\mathbb{R}^n\right).$$

We prove that the problem (1.1) has a unique solution $u \in W_p^l\left(\mathbb{R}^n\right)$ for $f \in L_p\left(\mathbb{R}^n\right)$ and the following uniform coercive estimate holds

$$|\alpha| \leq l|\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_{L_p\left(\mathbb{R}^n\right)} \leq C\|f\|_{L_p\left(\mathbb{R}^n\right)}. \tag{1.3}$$

Let O be a linear operator in L_p generated by problem (1.1), that is,

$$D(O) = W_p^l\left(\mathbb{R}^n\right), Ou = \sum_{|\alpha|\leq l} a_\alpha * D^\alpha u.$$

The estimate (1.3) implies that the operator O has a bounded inverse from $L_p\left(\mathbb{R}^n\right)$ into the Sobolev space $W_p^l\left(\mathbb{R}^n\right)$ which will be defined subsequently. Particularly, from the estimate (1.3), we obtain that O is a sectorial operator in $L_p\left(\mathbb{R}^n\right)$.

Let $L_{\mathbf{p}} = L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}\right)$ for $\mathbf{p} = (p, p_1)$ denotes the space of strongly measurable functions f defined on $\mathbb{R}_+^{1+n}$ equipped with the mixed norm (see, e.g., section 3.10.2 in [15])

$$\|f\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}\right)} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}_+} |f(t,x)|^{p_1} dt \right)^{\frac{p}{p_1}} dx \right)^{\frac{1}{p}} < \infty, p_1, p \in (1, \infty).$$

By using the coercive property of the given elliptic operator O, we prove the well posedness of the Cauchy problem for the corresponding nonlocal fractional parabolic differential equation

$$\partial_t u + \sum_{|\alpha|\leq l} a_\alpha * D^\alpha u = f(t,x), \ u(0,x) = 0, \tag{1.4}$$

In other words, we show that the problem (1.4) has a unique solution $u \in W_{\mathbf{p}}^{1,\gamma}\left(\mathbb{R}_+^2\right)$ for $f \in L_{\mathbf{p}}\left(\mathbb{R}_+^2\right)$ satisfying the following coercive estimate

$$\|\partial_t u\|_{L_{\mathbf{p}}}\left(\mathbb{R}_+^{n+1}\right) + \sum_{|\alpha|\le l} \|a_\alpha * D_x^\alpha u\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}\right)} + \|A * u\|_{L_{\mathbf{p}};\left(\mathbb{R}_+^{n+1}\right)} \le$$

$$M\|f\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}\right)}, \tag{1.5}$$

Let $\mathbb{C}$ denote the set of complex numbers and

$$S_\varphi = \{\lambda;\ \ \lambda \in \mathbb{C}, |\arg \lambda| \le \varphi\} \cup \{0\}, 0 \le \varphi < \pi.$$

Here, $B(E_1, E_2)$ denotes the space of bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ it denotes by $B(E)$. Let $D(A)$, $R(A)$ denote the domain and range of the linear operator in E, respectively. Let Ker A denote a null space of A. A closed linear operator A is said to be $\varphi-$ sectorial (or sectorial for $\varphi = 0$) in a Banach space E with bound $M > 0$ if Ker $A = \{0\}$, $D(A)$, and $R(A)$ are dense on E, and $\left\|(A + \lambda I)^{-1}\right\|_{B(E)} \le M|\lambda|^{-1}$ for all $\lambda \in S_\varphi$, $\varphi \in [0, \pi)$, where I is an identity operator in E. Sometimes $A + \lambda I$ will be written as $A + \lambda$ and will be denoted by A_λ. It is known [17, §1.15.1] that the powers A^θ, $\theta \in (-\infty, \infty)$ for a positive operator A exist.

A sectorial operator $A(x), x \in \mathbb{R}^n$ is said to be uniformly sectorial in a Banach space E if there exists a $\varphi \in [0, \pi)$ such that the uniformly estimate holds

$$\left\|(A(x) + \lambda I)^{-1}\right\|_{B(E)} \le M|\lambda|^{-1}$$

for all $\lambda \in S_\varphi$.

For any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i \in [0, \infty)$ the function $(i\xi)^\alpha$ will be defined as:

$$(i\xi)^\alpha = \begin{cases} (i\xi_1)^{\alpha_1} \ldots (i\xi_n)^{\alpha_n}, \xi_1\xi_2 \ldots \xi_n \ne 0 \\ 0, \xi_1, \xi_2 \ldots \xi_n = 0, \end{cases},$$

where

$$(i\xi_k)^{\alpha_k} = \exp\left[\alpha_k \left(\ln|\xi_k| + i\frac{\pi}{2} \operatorname{sgn} \xi_k\right)\right], k = 1, 2, \ldots, n.$$

The Liouville derivatives $D^\alpha u$ of the function u is defined as in [17]. Let $s \in \mathbb{R}$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$. Let F denotes the Fourier transform defined by

$$\hat{u}(\xi) = Fu = (2\pi)^{-\frac{n}{2}}_{\mathbb{R}^n} e^{-ix\xi} u(x) dx \text{ for } u \in S\left(\mathbb{R}^n\right) \text{ and } x, \xi \in \mathbb{R}^n.$$

Here, $S' = S'\left(\mathbb{R}^n\right)$ denotes the space of linear continuous mappings from $S\left(\mathbb{R}^n\right)$ into $\mathbb{C}$ and it is called the Schwartz distributions. Consider the following fractional Sobolev space (see, e.g., section 2.3 in [15])

$$W_p^s\left(\mathbb{R}^n\right) = \{u\ \ u \in S'\left(\mathbb{R}^n\right), F^{-1}\left(1 + |\xi|^2\right)^{\frac{s}{2}} Fu \in L_p\left(\mathbb{R}^n\right),$$

$$\left.\|u\|_{W_p^s\left(\mathbb{R}^n\right)} = \|u\|_{L_p\left(\mathbb{R}^n\right)} + \left\|F^{-1}\left(1 + |\xi|^2\right)^{\frac{s}{2}} Fu\right\|_{L_p\left(\mathbb{R}^n\right)} < \infty \right\}.$$

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_α.

The embedding theorems in the function spaces play a key role in the theory of ordinary and partial differential equations. From [11] we obtain the estimating lower order derivatives

Theorem A1 *Suppose* $1 < p \le q < \infty$ *and* $s \in (0, \infty)$ *with* $\varkappa = \frac{1}{s}\left[|\alpha| + n\left(\frac{1}{p} - \frac{1}{q}\right)\right] \le 1$, $0 \le \mu \le 1 - \varkappa$, *then the embedding*

$$D^\alpha W_p^s\left(\mathbb{R}^n\right) \subset L_q\left(\mathbb{R}^n\right)$$

is continuous and there exists a constant $C_\mu > 0$, *depending only on* μ *such that*

$$\|D^\alpha u\|_{L_q(\mathbb{R}^n)} \leq C_\mu \left[h^\mu \|u\|_{W_p^s(\mathbb{R}^n)} + h^{-(1-\mu)} \|u\|_{L_p(\mathbb{R}^n)} \right]$$

for all $u \in W_p^s(\mathbb{R}^n)$ *and* $0 < h \leq h_0 < \infty$.

2 | NONLOCAL FRACTIONAL ELLIPTIC EQUATION

Consider the problem (1.1).

Condition 2.1 Assume $a_\alpha \in L_\infty(\mathbb{R}^n)$ such that

$$L(\xi) = \sum_{|\alpha| \leq l} \hat{a}_\alpha(\xi)(i\xi)^\alpha \in S_{\varphi_1}, |L(\xi)| \geq M \sum_{k=1}^{n} \left|\hat{a}_{\alpha(l,k)}\right| |\xi_k|^l, \tag{2.1}$$

for

$$\alpha(l,k) = (0,0, \dots, l, 0, 0, \dots, 0), \text{i.e } \alpha_i = 0, i \neq k,$$

Consider the following operator functions

$$\sigma_1(\xi,\lambda) = \lambda\sigma_0(\xi,\lambda), \sigma_2(\xi,\lambda) = \sum_{|\alpha| \leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \hat{a}_\alpha(\xi)(i\xi)^\alpha \sigma_0(\xi,\lambda), \tag{2.2}$$

where

$$\sigma_0(\xi,\lambda) = [L(\xi)+\lambda]^{-1}.$$

Let

$$X = L_p(\mathbb{R}^n), Y = W_p^l(\mathbb{R}^n).$$

In this section we prove the following:

Theorem 2.1 *Assume that the* Condition 2.1 *is satisfied. Suppose that* $\gamma \in (1,2]$, and $\lambda \in S_{\varphi_2}$. *Then for* $f \in X$, $0 \leq \varphi_1 < \pi - \varphi_2$ *and* $\varphi_1 + \varphi_2 \leq \varphi$ *there is a unique solution* u *of the* Equation (1.1) *belonging to* Y *and the coercive uniform estimate holds*

$$|\alpha| \leq l |\lambda|^{1-\frac{|\alpha|}{l}} \|a * D^\alpha u\|_X + |\lambda| \|u\|_X \leq C\|f\|_X. \tag{2.3}$$

For the proving of Theorem 2.1 we need the following lemmas:

Lemma 2.1 *Assume* Condition 2.1 *holds and* $\lambda \in S_{\varphi_2}$ *with* $\varphi_2 \in [0,\pi)$, *where* $\varphi_1 + \varphi_2 < \pi$, *then the operator functions* $\sigma_i(\xi,\lambda)$ *are uniformly bounded, that is,*

$$|\sigma_i(\xi,\lambda)| \leq C, i = 0,1,2.$$

Proof. By virtue of [18] (lemma 2.3), for $L(\xi) \in S_{\varphi_1}$, $\lambda \in S_{\varphi_2}$ and $\varphi_1 + \varphi_2 < \pi$ there exists a positive constant C such that

$$|\lambda + L(\xi)| \geq C(|\lambda| + |L(\xi)|). \tag{2.4}$$

Since $L(\xi) \in S_{\varphi_1}$ in view of Condition 2.1 and (2.4) the function $\sigma_0(\xi,\lambda)$ is uniformly bounded for all $\xi \in \mathbb{R}^n$, $\lambda \in S_{\varphi_2}$, that is,

$$\sigma_0(\xi,\lambda) \leq (|\lambda| + |L(\xi)|)^{-1} \leq M_0.$$

Moreover, we have

$$|\sigma_1(\xi,\lambda)| \leq M|\lambda|(|\lambda| + |L(\xi)|)^{-1} \leq M_1.$$

Next, let us consider σ_2. It is clear to see that

$$|\sigma_2(\xi,\lambda)| \leq C \sum_{|\alpha|\leq l} |\lambda|_{k=1}^{n} \left[|\xi||\lambda|^{-\frac{1}{l}}\right]^{\alpha_k} |\sigma_0(\xi,\lambda)| . \tag{2.5}$$

By setting $y_k = \left(|\lambda|^{-\frac{1}{l}} |\xi_k|\right)^{\alpha_k}$ in the following well known inequality

$$y_1^{\alpha_1} y_2^{\alpha_2} \ \dots\ y_n^{\alpha_n} \leq C\left(1 + \sum_{k=1}^{n} y_k^l\right), y_k \geq 0, |\alpha| \leq l \tag{2.6}$$

we get

$$\|\sigma_2(\xi,\lambda)\|_{B(E)} \leq C \sum_{|\alpha|\leq l} |\lambda| \left[1 + \sum_{k=1}^{n} |\xi_k|^l |\lambda|^{-1}\right] |\lambda + L(\xi)|^{-1}.$$

Taking into account the Condition 2.1 and (2.5) and (2.6) we obtain

$$|\sigma_2(\xi,\lambda)| \leq C\left(|\lambda| + \sum_{k=1}^{n} |\xi_k|^l\right)(|\lambda| + |L(\xi)|)^{-1} \leq C.$$

■

Lemma 2.2 *Assume the* Condition 2.1 *holds. Suppose* $\widehat{a}_\alpha \in C^{(n)}\left(\mathbb{R}^n\right)$ *and*

$$|\xi|^{|\beta|} \left|D^\beta \widehat{a}_\alpha(\xi)\right| \leq C_1, \beta_k \in \{0,1\}, \xi \in \mathbb{R}^n \backslash \{0\}, 0 \leq |\beta| \leq n \tag{2.7}$$

Then, the operators $|\xi|^{|\beta|} D_\xi^\beta \sigma_i(\xi,\lambda)$, $i = 0, 1, 2$ are uniformly bounded.

Proof. Consider the term $|\xi|^{|\beta|} D_\xi^\beta \sigma_0(\xi,\lambda)$. By using the Condition 2.1 and the above estimates (2.4)–(2.6)

$$|\xi_k| \left|D_{\xi_k} \sigma_0(\xi,\lambda)\right| \leq$$

$$\left[|\xi_k| \left|\frac{\partial}{\partial \xi_k} \widehat{a}_\alpha(\xi)\right| + \alpha_k \left|\widehat{a}_\alpha(\xi)\right|\right] \left|_{k=1}^{n} (i\xi_k)^{\alpha_k}\right| \left|[L(\xi) + \lambda]^{-2}\right| < \infty.$$

It easy to see that the operator $|\xi|^\beta D^{|\beta|} \sigma_0(\xi,\lambda)$ contain the similar terms as in $|\xi_k| \left|D_{\xi_k} \sigma_0(\xi,\lambda)\right|$ for all $\beta_k \in \{0,1\}$. Hence, we get

$$|\xi|^{|\beta|} \left|D_\xi^\beta \sigma_0(\xi,\lambda)\right| < \infty.$$

■

In a similar way, by using the Condition 2.1 and in view of the estimates (2.4)–(2.7) we obtain

$$|\xi|^{|\beta|} \left|D_\xi^\beta \sigma_i(\xi,\lambda)\right| < \infty, i = 1, 2. \tag{2.8}$$

Proof of Theorem 2.1 By applying the Fourier transform to Equation (1.1), we get

$$\widehat{u}(\xi) = \sigma_0(\xi,\lambda)\widehat{f}(\xi), \sigma_0(\xi,\lambda) = [L(\xi) + \lambda]^{-1}. \tag{2.9}$$

Hence, the solution of (1.1) can be represented as $u(x) = F^{-1}\sigma_0(\xi,\lambda)\widehat{f}$ and by Lemma 2.1 there are positive constants C_1 and C_2 such that

$$C_1|\lambda|\|u\|_X \leq \left\|F^{-1}\left[\lambda\sigma_0(\xi,\lambda)\hat{f}\right]\right\|_X \leq C_2|\lambda|\|u\|_X,$$
$$C_1\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\|a_\alpha * D^\alpha u\|_X \leq \left\|F^{-1}\left[\sigma_2(\xi,\lambda)\hat{f}\right]\right\|_X \leq$$
$$C_2\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\|a_\alpha * D^\alpha u\|_X. \tag{2.10}$$

Therefore, it is sufficient to show that the operators $\sigma_i(\xi,\lambda)$ are multipliers in X. Indeed, by Lemma 2.2 and by virtue of Mikhlin multiplier theorem (see, e.g., section 2.2 in [15]), we get that $\sigma_i(\xi,\lambda)$ are uniform multipliers in X. So, we obtain the conclusion. ∎

Result 2.1 Theorem 2.1 implies that the O operator created by the (1.1) problem is separable in X, that is, for all $f \in X$ there is a unique solution $u \in Y$ of the problem (1.1), all terms of Equation (1.1) are also from X and there are positive constants C_1 and C_2 so that

$$C_1\|Ou\|_X \leq |\alpha| \leq l\,\|a_\alpha * D^\alpha u\|_X + \|u\|_X \leq C_2\|Ou\|_X.$$

Indeed, if we put $\lambda = 1$ in (2.3), by Theorem 2.1 we get the second inequality. So it is remain to prove the first estimate. The first inequality is equivalent to the following estimate

$$\sum_{|\alpha|\leq l}\left\|F^{-1}\hat{a}_\alpha(i\xi)^\alpha\hat{u}\right\|_X \leq \sum_{|\alpha|\leq l}\left\|F^{-1}\hat{a}_\alpha(i\xi)^\alpha\sigma_0(\xi,\lambda)\hat{f}(\xi)\right\|_X.$$

So, it suffices to show that the operator functions

$$\sigma_0(\xi,\lambda),\ \sum_{|\alpha|\leq l}\hat{a}_\alpha(i\xi)^\alpha\sigma_0(\xi,\lambda)$$

are uniform Fourier multipliers in X. This fact is proved in a similar way as the proof of Theorem 2.1.

From Theorem 2.1, we have:

Result 2.2 Assume all conditions of Theorem 2.1 hold. Then, for all $\lambda \in S_\varphi$ the resolvent of the operator O exists and the following sharp coercive uniform estimate holds

$$|\alpha| \leq l|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a \times D^\alpha(O+\lambda)^{-1}\right\|_{B(X)} + \left\|(O+\lambda)^{-1}\right\|_{B(X)} \leq C. \tag{2.11}$$

Indeed, we infer from Theorem 2.1 that the operator $O+\lambda$ has a bounded inverse from X to Y. So, the solution u of the Equation (1.1) can be expressed as $u(x) = (O+\lambda)^{-1}f$ for all $f \in X$. Then estimate (2.4) implies the estimate (2.11).

Theorem 2.2 *Assume that the* Condition 2.1 *is satisfied. Suppose that* $\gamma \in (1,2]$, *and* $\lambda \in S_{\varphi_2}$. *Then for* $f \in X$, $0 \leq \varphi_1 < \pi - \varphi_2$ *and* $\varphi_1 + \varphi_2 \leq \varphi$ *there is a unique solution* u *of the* Equation (1.1) *belonging to* Y *and the following coercive uniform estimate holds*

$$|\alpha| \leq l|\lambda|^{1-\frac{|\alpha|}{l}}\|D^\alpha u\|_X + \|u\|_X \leq C\|f\|_X. \tag{2.12}$$

Proof. The estimate (2.12) is derived by reasoning as in Theorem 2.2. ∎

From Theorem 2.2, we have the following results:

Result 2.3 There are positive constants C_1 and C_2 so that

$$C_1\|Ou\|_X \leq |\alpha| \leq l\,\|D^\alpha u\|_X + \|Au\|_X \leq C_2\|Ou\|_X. \tag{2.13}$$

From Theorem 2.2, we obtain

Result 2.4 Assume all conditions of Theorem 2.2 hold. Then, for all $\lambda \in S_\varphi$ the resolvent of operator O exists and the following sharp uniform estimate holds

$$|\alpha| \leq l|\lambda|^{1-\frac{|\alpha|}{l}} \left\|D^\alpha (O+\lambda)^{-1}\right\|_{B(X)} + \left\|(O+\lambda)^{-1}\right\|_{B(X)} \leq C. \tag{2.14}$$

Result 2.5 Theorem 2.2 particularly implies that the operator O is sectorial in X. Then the operators O^s are generators of analytic semigroups in X for $s \leq \frac{1}{2}$ (see, e.g., section 1.14.5 in [15]).

Example 2.1 Let we put $n = 2$, $\alpha_1 = \alpha_2 = \alpha \in (1,2)$, and $l = 2\alpha$ in (1.1). Then we have the following fractional partial differential equation

$$a_{11} * D^{2\alpha}_{x_1} u + a_{12} * D^{\alpha}_{x_1} D^{\alpha}_{x_2} u + a_{22} * D^{2\alpha}_{x_1} u + b_1 * D^{\alpha}_{x_1} u + b_2 * D^{\alpha}_{x_2} u + \lambda u = f(x), \tag{2.15}$$

where $a_{ij} = a_{ij}(x)$, $b_j = b_j(x)$ are complex valued functions, λ is a complex parameter and $x = (x_1, x_2) \in \mathbb{R}^2$.

We assume that $a_{ij}, b_j \in L_\infty\left(\mathbb{R}^n\right)$ such that

$$L(\xi) = \hat{a}_{11}(\xi)(i\xi_1)^{2\alpha} + \hat{a}_{12}(\xi)(i\xi_1)^{\alpha}(i\xi_2)^{\alpha} + \hat{a}_{22}(\xi)(i\xi_2)^{2\alpha} \in S_{\varphi_1},$$

$$|L(\xi)| \geq M\left(\left|\hat{a}_{11}(\xi)\right| |\xi_1|^{2\alpha} + \left|\hat{a}_{22}(\xi)\right| |\xi_2|^{2\alpha}\right) \text{ for all } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Moreover, let $\lambda \in S_{\varphi_2}$ such that $\varphi_1 + \varphi_2 < \pi$. Then from Theorem 2.1 we obtain the following:

For all $f \in L^p\left(\mathbb{R}^2\right)$ there is a unique solution u of the Equation (2.15) belonging to $W^{2\alpha}_p\left(\mathbb{R}^2\right)$ and the following coercive uniform estimate holds

$$\left\|a_{11} * D^{2\alpha}_{x_1} u\right\|_{L^p\left(\mathbb{R}^2\right)} + |\lambda|^{\frac{1}{2}} \left\|b_1 * D^{\alpha}_{x_1} u\right\|_{L^p\left(\mathbb{R}^2\right)} + \left\|a_{12} * D^{\alpha}_{x_1} D^{\alpha}_{x_2} u\right\|_{L^p\left(\mathbb{R}^2\right)} +$$

$$\left\|a_{22} * D^{2\alpha}_{x_2} u\right\|_{L^p\left(\mathbb{R}^2\right)} + |\lambda|^{\frac{1}{2}} \left\|b_2 * D^{\alpha}_{x_2} u\right\|_{L^p\left(\mathbb{R}^2\right)} + |\lambda|\|u\|_{L^p\left(\mathbb{R}^2\right)} \leq C\|f\|_{L^p\left(\mathbb{R}^2\right)}.$$

3 | THE CAUCHY PROBLEM FOR FRACTIONAL PARABOLIC EQUATION

In this section, we shall consider the following Cauchy problem for the parabolic FDOE

$$\frac{\partial u}{\partial t} + \sum_{|\alpha| \leq l} a_\alpha * D^\alpha u = f(t,x), u(0,x) = 0, t \in \mathbb{R}_+, x \in \mathbb{R}^n, \tag{3.1}$$

where a is a complex number, D^α_x is a fractional derivative in x for $\alpha_k \in (1,2]$ defined by (1.2).

By applying Theorem 2.1 we establish the maximal regularity of the problem (3.1) in mixed $L_{\mathbf{p}}$ spaces, where $\mathbf{p} = (p_1, p)$. Let O denote the operator generated by problem (1.1) for $\lambda = 0$. Here, $Z = L_{\mathbf{p}}\left(\mathbb{R}^{n+1}_+\right)$ for $\mathbf{p} = (p, p_1)$ will denote the space of all $\mathbf{p}$-summable complex valued functions on $\mathbb{R}^{n+1}_+$ with mixed norm, that is, the space of all measurable complex-valued functions f defined on $\mathbb{R}^{n+1}_+$ for which

$$\|f\|_{L_{\mathbf{p}}\left(\mathbb{R}_{+}^{n+1};H\right)}=\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}_+}|f(t,x)|^p dx\right)^{\frac{p_1}{p}} dt\right)^{\frac{1}{p_1}}<\infty.$$

Let $Z^{1,\gamma}=W_{\mathbf{p}}^{1,l}\left(\mathbb{R}_{+}^{n+1}\right)$ denotes the space of all functions $u\in L_{\mathbf{p}}\left(\mathbb{R}_{+}^{n+1}\right)$ possessing the generalized derivative $D_t u=\frac{\partial u}{\partial t}\in Z$ and fractional derivatives $D_x^{\alpha}u\in Z$ with respect to x for $|\alpha|\leq l$ with the norm

$$\|u\|_{Z^{1,2}(A)}=\|u\|_Z+\|\partial_t u\|_Z+\left\|D_x^{\gamma}u\right\|_Z,$$

where $u=u(t,x)$.

Now, we are ready to state the main result of this section.

Theorem 3.1 *Assume the conditions of* Theorem 2.1 *hold for* $\varphi\in\left(\frac{\pi}{2},\pi\right)$. *Then for* $f\in Z$ *the problem* (3.1) *has a unique solution* $u\in Z^{1,\gamma}(A)$ *satisfying the following uniform coercive estimate*

$$\|\partial_t u\|_Z+\sum_{|\alpha|\leq l}\|a_\alpha * D^{\alpha}u\|_Z+\|u\|_Z\leq C\|f\|_Z.$$

Proof. By definition of $X=L_p\left(\mathbb{R}^n\right)$ and mixed space $L_{\mathbf{p}}\left(\mathbb{R}_{+}^{n+1}\right)$, $\mathbf{p}=(p,p_1)$, we have

$$\|u\|_{L_{p_1}}(0,\infty)=\left(\int_0^\infty\|u(t)\|_X^{p_1}dt\right)^{\frac{1}{p_1}}=\left(\int_0^\infty\|u(t)\|_{L_p\left(\mathbb{R}^n\right)}^{p_1}dt\right)^{\frac{1}{p_1}}=\|u\|_Z.$$

Therefore, the problem (3.1) can be expressed as the following Cauchy problem for the abstract parabolic equation

$$\frac{du}{dt}+Ou(t)=f(t),u(0)=0,t\in(0,\infty). \tag{3.2}$$

Then, by virtue of in [16] (theorem 4.2), we obtain that for $f\in L_{p_1}(0,\infty;X)$ the problem (3.2) has a unique solution $u\in W_{p_1}^1(0,\infty;D(O),X)$ satisfying the following estimate

$$\left\|\frac{du}{dt}\right\|_{L_{p_1}(0,\infty;X)}+\|Ou\|_{L_{p_1}(0,\infty;X)}\leq C\|f\|_{L_{p_1}(0,\infty;X)}.$$

From the Theorem 2.2, relation (3.2) and from the above estimate we get the conclusion. ∎

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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