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The regularity properties and blow-up of the solutions for nonlocal general wave equations

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Abstract

In this paper, the Cauchy problem for nonlocal linear and nonlinear wave equations are studied. The equations include the general differential operators. The existence, uniqueness, L^p-regularity properties and blow-up at finite time of solutions of the Cauchy problem is obtained. By choosing differential operators including in equations, the regularity properties of a different type wave equations are studied.

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1. Introduction

The aim in this paper is to study the local existence and uniqueness of solution of the Cauchy problem for the following wave equation

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$$L_0 u_{tt} + L_1 u = L_2 f(u), \tag{1.1}$$

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x), x \in \mathbb{R}^n, t \in (0,T), \tag{1.2}$$

where L_i are nonlocal differential operators with variable coefficients defined by

$$L_i u = \sum_{|\alpha| \le m_i} a_{i\alpha} * D^\alpha u, i = 0, 1, 2, u \in W_p^m\left(\mathbb{R}^n\right), m = \max\{m_0, m_1, m_2, \} \tag{1.0}$$

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \ |\alpha| = \sum_{k=1}^n \alpha_k, D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha n}},$$

$a_{i\alpha}$ are complex valued functions on $\mathbb{R}^n$, $f(u) = f(x,t,u)$ is a given nonlinear function, $\varphi(x)$, $\psi(x)$ are given initial value functions and $u(x,t)$ is a complex-valued unknown function. Here, $u * \upsilon$ denote the convolution of functions u and υ.

Note that, the predictions of classical elasticity theory become inaccurate when the characteristic length of an elasticity problem is comparable to the atomic length scale. To fix this situation, a nonlocal theory of elasticity was introduced (see [1–3] and the references cited therein) and the main feature of the new theory is the fact that its predictions were more down to earth than those of the classical theory. For other generalizations of elasticity we refer the reader to [4–14]. Regularity properties of nonlocal PDEs were studied e.g. in [15–21]. Moreover the regularity properties nonlocal partial differential equations were investigated e.g. in [22–25].

In this paper have been found sufficient conditions that depend on the nature and mutual relevance of the differential operators included in the equation to ensure that there exists a unique solution of the problem, being L^p regular and blow up in finite time. By choosing the operators L_i, we obtain a different class of wave equations which occur in a wide variety of physical systems, such as in the area of such as the peridynamical theory of continuum mechanics, nonlocal wave propagation, and the modeling of nonlocal diffusion processes, the propagation of longitudinal deformation waves in an elastic rod, hydro-dynamical process in plasma and in materials science.

We think that this article is very interesting in the L^p regularity theory for wave equations. Here, the Cauchy problem for linear and nonlinear wave equations with differential operator coefficients are studied. Here, comprehensively for the first time, the existence, uniqueness, L^p regularity and blow up properties at finite time of solution of the Cauchy problem for these equations are established. Moreover, the method of proofs naturally differs to those used in previous works. Since the problem includes general differential operators in leading part we need same extra mathematics tools for deriving considered conclusions. By this reason, in the proof we use modern analysis tools as: Fourier multiplier theorems in L_p spaces, embedding of Sobolev and Besov spaces, theory of semigrops of linear operators in Banach spaces, interpolation of Banach spaces, and etc.

For example, if we choose $L_1 = L_2 = -\Delta$, where Δ-is 3-dimensional Laplace operator, we obtain the existence, uniqueness, regularity properties and blow-up in finite time of solutions of the following Cauchy problem

$$u_{tt} - \Delta * u = \Delta * f(u), x \in \mathbb{R}^3, t \in (0,T), \tag{1.3}$$

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x). \tag{1.4}$$

Let

$$L_i u = \sum_{|\alpha|\leq 4} a_{i\alpha}(x) * D^{\alpha} u,\ i = 0, 1,\ 2,\ u \in W_p^4(\mathbb{R}^n),$$

where $a_{i\alpha}$ are complex valued functions, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, α_k are natural numbers and $|\alpha| = \sum_{k=1}^{3} \alpha_k$.

We obtain the existence, uniqueness, regularity properties and blow-up in finite time of solutions of the following Cauchy problem

$$L_0 u_{tt} + L_1 u = L_2 f(u)(x,t), \tag{1.5}$$
$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ x \in \mathbb{R}^3,\ t \in (0,T),$$

where

$$\varphi, \psi \in W_p^s\left(\mathbb{R}^3\right),\ s > \frac{3}{p},\ \ p \in [1, \infty].$$

Let now

$$L_0 u = \sum_{|\alpha|\leq 2} a_{0\alpha}(x) * D^{\alpha} u,\ \ L_1 u = \sum_{|\alpha|\leq 2} a_{1\alpha}(x) * D^{\alpha} u,$$
$$L_2 u = \sum_{|\alpha|\leq 4} a_{2\alpha}(x) * D^{\alpha} u,\ u \in W_p^4(\mathbb{R}^n),$$

where $a_{\alpha i}$ are complex valued functions, $\alpha = (\alpha_1, \alpha_2)$, α_k are natural numbers and $|\alpha| = \alpha_1 + \alpha_2$.

Consider the Cauchy problem for the following wave equation

$$L_0 u_{tt} + L_1 u = L_2 f(u)(x,t),\ x \in \mathbb{R}^n,\ t \in (0,T), \tag{1.6}$$
$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ t \in (0,T)\ \ x \in \mathbb{R}^2,\ t \in (0,T),$$

where

$$\varphi,\ \psi \in W^{s,p}\left(\mathbb{R}^3\right),\ s > \frac{3}{p},\ p \in [1, \infty].$$

By using the general result for (1.1)–(1.2), we obtain the existence, uniqueness, L^p regularity and blow up properties (in finite time) of solution of the Cauchy problem of (1.6).

In this paper, we obtain the existence and uniqueness of solution and L_p-regularity properties of the problem (1.1)–(1.2). The strategy is to express the Boussinesq equation as an integral equation. To treat the nonlinearity as a small perturbation of the linear part of the equation, the contraction mapping theorem is used. Also, a priori estimates on L^p norm of solutions of the linearized version are utilized. The key step is the derivation of the uniform estimate of the solutions of the linearized wave equation. The methods of harmonic analysis, operator theory, interpolation of Banach Spaces and embedding theorems in Sobolev spaces are the main tools implemented to carry out the analysis.

2. Definitions and background

In order to state our results precisely, we introduce some notations and some function spaces.

Let E be a Banach space. $L_p(\Omega; E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_{L_p} = \|f\|_{L_p(\Omega;E)} = \left(\int_\Omega \|f(x)\|_E^p \, dx \right)^{\frac{1}{p}}, 1 \leq p < \infty,$$

$$\|f\|_{L_\infty(\Omega:E)} = \operatorname*{ess\,sup}_{x \in \Omega} \|f(x)\|_E .$$

Let $\mathbb{C}$ denote the set of complex numbers. For $E = \mathbb{C}$ the $L_p(\Omega; E)$ denotes by $L_p(\Omega)$.

Let m be a positive integer. $W_p^m(\Omega)$ denotes the Sobolev space, i.e. space of all functions $u \in L_p(\Omega)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L_p(\Omega)$, $1 \leq p \leq \infty$ with the norm

$$\|u\|_{W_p^m(\Omega)} = \|u\|_{L_p(\Omega)} + \sum_{k=1}^{n} \left\| \frac{\partial^m u}{\partial x_k^m} \right\|_{L_p(\Omega)} < \infty.$$

Let $S(\mathbb{R}^n)$ denote Schwartz class, i.e., the space of rapidly decreasing smooth functions on $\mathbb{R}^n$, equipped with its usual topology generated by seminorms. Let $S'(\mathbb{R}^n)$ denote the space of all continuous linear operators $L : S(\mathbb{R}^n) \to \mathbb{C}$, equipped with the bounded convergence topology. Recall $S(\mathbb{R}^n)$ is norm dense in $L_p(\mathbb{R}^n)$ when $1 \leq p < \infty$. Let F denote the Fourier transform defined by

$$\hat{u}(\xi) = Fu = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\xi} u(x) \, dx \text{ for } u \in S(\mathbb{R}^n) \text{ and } x, \xi \in \mathbb{R}^n.$$

Let $L_p^s(\mathbb{R}^n)$, $-\infty < s < \infty$ denotes Liouville-Sobolev space of order s which is defined as:

$$L_p^s = L_p^s\left(\mathbb{R}^n\right) = (I - \Delta)^{-\frac{s}{2}} L_p\left(\mathbb{R}^n\right)$$

with the norm

$$\|u\|_{L_p^s} = \left\| (I - \Delta)^{\frac{s}{2}} u \right\|_{L_p(\mathbb{R}^n)} = \left\| \left(1 + |\xi|^2\right)^{\frac{s}{2}} \hat{u} \right\|_{L_p(\mathbb{R}^n)} < \infty.$$

It is clear that $L_p^0(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. It is known that $L_p^m(\mathbb{R}^n) = W_p^m(\mathbb{R}^n)$ for the positive integer m (see e.g. [26, § 2.3]).

Let $\mathbb{R}_T^n = \mathbb{R}^n \times (0, T)$. In a similar way, we define the following anisotropic Sobolev space:

$$W_p^{2,s}\left(\mathbb{R}_T^n\right) = \left\{u \in L^p\left(\mathbb{R}_T^n\right),\ \partial_t^2 u \in L^p\left(\mathbb{R}_T^n\right),\right.$$

$$\mathbb{F}_x^{-1}\left(I + |\xi|^2\right)^{\frac{s}{2}} \hat{u} \in L^p\left(\mathbb{R}_T^n\right),\ \|u\|_{W_p^{2,s,}\left(\mathbb{R}_T^n\right)} =$$

$$\left.\|u\|_{L^p(\mathbb{R}^n_T)} + \left\|\partial_t^2 u\right\|_{L^p(\mathbb{R}^n_T)} + \left\|\mathbb{F}_x^{-1}\left(I+|\xi|^2\right)^{\frac{s}{2}}\hat{u}\right\|_{L^p(\mathbb{R}^n_T)} < \infty\right\}.$$

Let $1 \leq p \leq q < \infty$. *A* function $\Psi \in L_\infty(\mathbb{R}^n)$ is called a Fourier multiplier from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if the map $u \to F^{-1}\Psi(\xi)Fu$ for $u \in S(\mathbb{R}^n)$ is well defined and extends to a bounded linear operator

$$T : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n).$$

Let $L_q^*(E)$ denote the space of all $E-$valued function space such that

$$\|u\|_{L_q^*(E)} = \left(\int_0^\infty \|u(t)\|_E^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty,\ 1 \leq q < \infty,\ \|u\|_{L_\infty^*(E)} = \sup_{t\in(0,\infty)} \|u(t)\|_E .$$

Here, F denotes the Fourier transform. Fourier-analytic representation of Besov space on $\mathbb{R}^n$ is defined as:

$$B^s_{p,q}\left(\mathbb{R}^n\right) = \left\{u \in S'\left(\mathbb{R}^n\right),\right.$$

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n)} = \left\|F^{-1}t^{\varkappa-s}\left(1+|\xi|^{\frac{\varkappa}{2}}\right)e^{-t|\xi|^2}Fu\right\|_{L_q^*(L_p(\mathbb{R}^n))},$$

$$|\xi|^2 = \sum_{k=1}^n \xi_k^2,\ \xi = (\xi_1, \xi_2, ..., \xi_n),\ p \in (1,\infty),\ q \in [1,\infty],\ \varkappa > s\}.$$

It should be noted that, the norm of Besov space does not depend on $\varkappa$ (see e.g. [26, § 2.3]). For $p = q$ the space $B^s_{p,q}(\mathbb{R}^n)$ will be denoted by $B^s_p(\mathbb{R}^n)$.

Let E_1 and E_2 be two Banach spaces. $(E_1, E_2)_{\theta,p}$ for $\theta \in (0,1)$, $p \in [1,\infty]$ denotes the real interpolation spaces defined by K-method [26, §1.3.2]. Let E_1 and E_2 be two Banach spaces. $B(E_1, E_2)$ will denote the space of all bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ it will be denoted by $B(E)$.

Here,

$$X_p = L^p\left(\mathbb{R}^n\right),\ 1 \leq p \leq \infty,\ Y^{s,p} = L^s_p\left(\mathbb{R}^n\right),$$

$$Y_1^{s,p} = L_p^s\left(\mathbb{R}^n\right) \cap L_1\left(\mathbb{R}^n\right),\ Y_\infty^{s,p} = L_p^s\left(\mathbb{R}^n\right) \cap L_\infty\left(\mathbb{R}^n\right),$$

$$Y_\infty^{s,2,p} = W_p^{2,s}\left(\mathbb{R}^n_T\right) \cap X_\infty,$$

$$Q = Q(\xi) = L_1(\xi)L_0^{-1}(\xi),\ L(\xi) = L_2(\xi)L_0^{-1}(\xi).$$

Definition 2.1. For any $T > 0$ the function $u \in C^2\left([0\ ,T);Y_\infty^{2,s,p}\right)$ satisfies the equation (1.1)–(1.2) a.e. in $\mathbb{R}^n_T$ is called the continuous solution or the strong solution of the problem (1.1)–(1.2). If $T < \infty$, then $u(x,t)$ is called the local strong solution of the problem (1.1)–(1.2). If $T = \infty$, then $u(x,t)$ is called the global strong solution of (1.1)–(1.2).

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_α. Moreover, for u, $\upsilon > 0$ the relation $u \lesssim \upsilon$, $u \approx \upsilon$ means that there exist positive constants C, C_1, C_2 independent on u and υ such that, respectively

$$u \leq C\upsilon,\ C_1\upsilon \leq u \leq C_2\upsilon.$$

The paper is organized as follows: In Section 2 some definitions and background are given. Section 3 we obtain the existence of unique solution and a priory estimates for solution of the linearized problem (1.1)–(1.2). In Section 4 we show the existence and uniqueness of local strong solution of the problem (1.1)–(1.2). Section 5 devote to global existence of solution of the problem (1.1)–(1.2). In section 6 blow-up in finite time of solutions of the Cauchy problem (1.1)–(1.2) is obtained. Finally, in Section 7 as an application by choosing the concrete differential operators L_i, $i = 0$, 1, 2, the existence, uniqueness, L^p-regularity properties and blow-up of solutions of the different Cauchy problem are derived.

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say h, we write C_h.

3. Estimates for linearized equation

In this section, we make the necessary estimates for solutions of initial value problems for the linearized general wave equation

$$L_0 u_{tt} + L_1 u = L_2 g\left(x, t\right), x \in \mathbb{R}^n, t \in \left(0, T\right), \tag{3.1}$$

$$u\left(x, 0\right) = \varphi\left(x\right), u_t\left(x, 0\right) = \psi\left(x\right), x \in \mathbb{R}^n, t \in \left(0, T\right),$$

where L_i are differential operators defined by (1.0).

Let

$$L_i = L_i\left(\xi\right) = \sum_{|\alpha| \leq m_i} \hat{a}_{i\alpha}\left(\xi\right)\left(i\xi_1\right)^{\alpha_1}\left(i\xi_2\right)^{\alpha_2} \ldots \left(i\xi_n\right)^{\alpha_n}, i = 0, 1, 2, \tag{3.0}$$

where $\hat{a}_{i\alpha}\left(\xi\right)$ are the Fourier transform of $a_{i\alpha}\left(x\right)$. Let

$$\beta = \left(\beta_1, \beta_2, \ldots, \beta_n\right), D^\alpha = D_\xi^\alpha = \frac{\partial^{|\beta|}}{\partial\xi_1^{\beta_1}\partial\xi_2^{\beta_2} \ldots \partial\xi_n^{\beta n}},$$

$$m = \max\left\{m_0, m_1, m_2\right\}, Q = Q\left(\xi\right) = L_1\left(\xi\right) L_0^{-1}\left(\xi\right), L\left(\xi\right) = L_2\left(\xi\right) L_0^{-1}\left(\xi\right). \tag{3.2}$$

Condition 3.1. Let $L_0\left(\xi\right) \neq 0$ and there exists a constant $M_0 > 0$ such that $\left|L_0\left(\xi\right)\right| \geq M_0\left|\xi\right|^{m_0}$ for all $\xi \in \mathbb{R}^n$. Let $a_{i\alpha}\left(.\right) \in L_1\left(\mathbb{R}^n\right)$. Assume that there exist positive constants M_1 and M_2 depend only on $a_{\imath\alpha}$ such that

$$\left|Q^{-\frac{1}{2}}\left(\xi\right)\right| \leq M_1\left(1 + |\xi|^2\right)^{\frac{\delta}{2}}, \left|L\left(\xi\right) Q^{-\frac{1}{2}}\left(\xi\right)\right| \leq M_2\left(1 + |\xi|^2\right)^{\frac{\delta}{2}} \tag{3.3}$$

for all $\xi \in \mathbb{R}^n$, where

$$\delta = \frac{s-\sigma-n}{q}, \sigma > n\left(\frac{1}{q}+\frac{1}{p}\right), s > \sigma + n \text{ with a } p \in [1,\infty] \text{ and a } q \in [1,2]\,.$$

Remark 3.1. The Condition 3.1 means that $L_0^{-1}(\xi)$ is uniformly bounded. Moreover, if

$$(m_1 - m_0) \leq 2\delta,\ (m_2 - m_0)(m_1 - m_0)^{\frac{1}{2}} \leq \delta,$$

i.e. (3.3) are hold trivially if $m_0 = m_1 = m_2$.

First we need the following lemmas

Lemma 3.1. *Suppose that* $Q(\xi) \neq 0$ *for each* $\xi \in \mathbb{R}^n$. *Then problem* (3.1) *has a generalized solution.*

Proof. By using of Fourier transform we get from (3.1):

$$\hat{u}_{tt}(\xi,t) + Q(\xi)\hat{u}(\xi,t) = L(\xi)\hat{g}(\xi,t), \tag{3.4}$$
$$\hat{u}(\xi,0) = \hat{\varphi}(\xi),\, \hat{u}_t(\xi,0) = \hat{\psi}(\xi),\, \xi \in \mathbb{R}^n, t \in (0,T),$$

where $\hat{u}(\xi,t)$ is a Fourier transform of $u(x,t)$ with respect to x. By using the variation of constants we get that there exists a solution of the problem (3.4) that can be written as:

$$\hat{u}(\xi,t) = C(\xi,t)\hat{\varphi}(\xi) + S(\xi,t)\hat{\psi}(\xi) + \int_0^t \hat{\Phi}(\xi,t-\tau)\hat{g}(\xi,\tau)\,d\tau, t \in (0,T), \tag{3.5}$$

here, $C(\xi,t)$, $S(\xi,t)$ cosine and sine functions generated by $Q = Q(\xi)$ (see e.g. [27, § 11]), i.e.

$$C(\xi,t) = \cos Q^{\frac{1}{2}}t = \frac{e^{iQ^{\frac{1}{2}}t} + e^{-iQ^{\frac{1}{2}}t}}{2},\ S(\xi,t) = Q^{-\frac{1}{2}}\sin Q^{\frac{1}{2}}t =$$
$$Q^{-\frac{1}{2}}\frac{e^{iQ^{\frac{1}{2}}t} - e^{-iQ^{\frac{1}{2}}t}}{2i}, \hat{\Phi}(\xi,t) = L(\xi)Q^{-\frac{1}{2}}(\xi)\sin\left(Q^{\frac{1}{2}}t\right).$$

From (3.5) we get that the solution of (3.1) can be expressed as

$$u(x,t) = S_1(t)\varphi(x) + S_2(t)\psi(x) + \tag{3.6}$$
$$\int_0^t F^{-1}\left[\hat{\Phi}(\xi,t-\tau)\hat{g}(\xi,\tau)\right]d\tau d\xi,\ t \in (0,T),$$

where F^{-1} denotes the inverse Fourier transformation, $S_1(t)$ and $S_2(t)$ are linear operators in X_p defined by

$$S_1(t)\varphi=(2\pi)^{-\frac{n}{2}}\int\limits_{\mathbb{R}^n}e^{ix\xi}C(\xi,t)\hat{\varphi}(\xi)\,d\xi,\ S_2(t)\psi=(2\pi)^{-\frac{n}{2}}\int\limits_{\mathbb{R}^n}e^{ix\xi}S(\xi,t)\hat{\psi}(\xi)\,d\xi.\quad\square \tag{3.7}$$

Theorem 3.1. *Assume the Condition 3.1 holds. Let* $\varphi,\ \psi,\ g(.,t)\in Y_1^{s,p}$ *for* $t\in(0,T)$ *and* $g(x,.)\in L_1(0,T)$ *for* $x\in\mathbb{R}^n$. *Then there exists a unique solution of the problem (3.1) satisfies the following estimate*

$$\|L_1*u\|_{X_\infty}+\|L_1*u_t\|_{X_\infty}\le C\left\{\|\varphi\|_{Y^{s,p}}+\|\varphi\|_{X_1}+\right. \tag{3.8}$$
$$\left.\|\psi\|_{Y^{s,p}}+\|\psi\|_{X_1}+\int\limits_0^t\left[\|g(.,\tau)\|_{Y^{s,p}}+\|g(.,\tau)\|_{X_1}\right]d\tau\right\}$$

uniformly with respect to $t\in[0,T]$.

Proof. Let $N\in\mathbb{N}$ and

$$\Pi_N=\left\{\xi:\xi\in\mathbb{R}^n,\ |\xi|\le N\right\},\ \Pi'_N=\left\{\xi:\xi\in\mathbb{R}^n,\ |\xi|\ge N\right\}.$$

It is clear to see that

$$\left\|F^{-1}C(\xi,t)\hat{\varphi}(\xi)\right\|_{X_\infty}+\left\|F^{-1}S(\xi)\hat{\psi}(\xi,t)\right\|_{X_\infty}\le$$
$$\left\|\int\limits_{\mathbb{R}^n}e^{ix\xi}C(\xi,t)\hat{\varphi}(\xi)\,d\xi\right\|_{L_\infty(\Pi_N)}+\left\|\int\limits_{\mathbb{R}^n}e^{ix\xi}S(\xi,t)\hat{\psi}(\xi)\,d\xi\right\|_{L_\infty(\Pi_N)}+ \tag{3.9}$$
$$\left\|F^{-1}C(\xi,t)\hat{\varphi}(\xi)\right\|_{L_\infty(\Pi'_N)}+\left\|F^{-1}S(\xi,t)\hat{\psi}(\xi)\right\|_{L\infty(\Pi'_N)}.$$

Using the Minkowski's inequality for integrals and uniformly boundedness of $C(\xi,t)$, $S(\xi,t)$ on Π_N we have

$$\left\|\int\limits_{\mathbb{R}^n}e^{ix\xi}C(\xi,t)\hat{\varphi}(\xi)\,d\xi\right\|_{L_\infty(\Pi_N)}+\left\|\int\limits_{\mathbb{R}^n}e^{ix\xi}S(\xi,t)\hat{\psi}(\xi)\,d\xi\right\|_{L_\infty(\Pi_N)}\lesssim \tag{3.10}$$
$$\left[\|\varphi\|_{X_1}+\|\psi\|_{X_1}\right].$$

Here,

$$G_1(\xi,t)=\left(1+|\xi|^2\right)^{-\frac{s}{2}}C(\xi,t),\ G_2(\xi,t)=\left(1+|\xi|^2\right)^{-\frac{s}{2}}S(\xi,t),$$

$$G_3(\xi,t)=\left(1+|\xi|^2\right)^{-\frac{s}{2}}L(\xi)\,S(\xi,t).$$

Let we show that

$$G_j(.,t) \in B_{q,1}^{n\left(\frac{1}{q}+\frac{1}{p}\right)}\left(\mathbb{R}^n; B\left(L_p, L_\infty\right)\right)$$

for all $t \in [0,T]$. By virtue of growth assumptions on polynomials $L_i(\xi)$ in Condition 3.1, we have

$$\sup_{t\in[0,T]} \left\|G_j(.,t)\right\|_{B\left(L_p,L_\infty\right)} \leq C,\ j=1,2,3. \tag{3.11}$$

Moreover, by embedding properties of Sobolev and Besov spaces it sufficient to show that

$$G_j(.,t) \in W_q^\sigma\left(\mathbb{R}^n; B\left(L_p, L_\infty\right)\right) \tag{3.12}$$

for $\sigma > n\left(\frac{1}{q}+\frac{1}{p}\right)$ and for some $q \in [1,2]$. For deriving (3.12) it sufficient to show

$$\left(1+|\xi|^2\right)^{\frac{\sigma}{2}} G_j(.,t) \in L_q\left(\mathbb{R}^n\right) \text{ for al } t \in [0,T].$$

Indeed, in view of (3.11), $\left(1+|\xi|^2\right)^{\frac{\sigma}{2}} G_j(\xi)$ are uniformly bounded for $\xi \in \mathbb{R}^n$. By virtue Condition 3.1 and by assumption (3.3), we have

$$\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{\frac{\sigma}{2}q} \left|G_j(\xi,t)\right|^q d\xi \lesssim \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\left(\frac{s-\sigma}{2}\right)q} d\xi < \infty.$$

Hence, by Fourier multiplier theorems (see e.g. [28, Theorem 4.3]) we get that the functions $G_j(\xi,t)$ are $L_p(\mathbb{R}^n) \to L_\infty(\mathbb{R}^n)$ Fourier multipliers. Hence, by Minkowski's inequality for integrals from (3.9)–(3.10) we obtain

$$\left\|F^{-1}C(\xi,t)\hat{\varphi}(\xi)\right\|_{L_\infty(\mathbb{R}^n)} + \left\|F^{-1}S(\xi,t)\hat{\psi}(\xi)\right\|_{L_\infty(\mathbb{R}^n)} \leq \tag{3.13}$$

$$C\left[\|\varphi\|_{Y^{s,p}} + \|\psi\|_{Y^{s,p}}\right].$$

By reasoning as the above, we have

$$\left\|F^{-1}\int_0^t \hat{\Phi}(\xi,t-\tau)\hat{g}(\xi,\tau)\,d\tau\right\|_{X_\infty} \leq . \tag{3.14}$$

$$C\int_0^t \left(\|g(.,\tau)\|_{Y^{s,p}} + \|g(.,\tau)\|_{X_1}\right)d\tau$$

Hence, from (3.13)–(3.14) we obtain

$$\|L_1 * u\|_{X_\infty} \leq C\left[\|\varphi\|_{Y^{s,p}} + \|\varphi\|_{X_1} + \right. \tag{3.15}$$

$$\|\psi\|_{Y^{s,p}}+\|\psi\|_{X_1}+\int_0^t\left(\|g(.,\tau)\|_{Y^{s,p}}+\|g(.,\tau)\|_{X_1}\right)d\tau\Bigg].$$

By differentiating from (3.4), we get

$$\hat{u}_t(\xi,t)=-Q(\xi)S(t)\hat{\varphi}(\xi)+C(t)\hat{\psi}(\xi)+ \tag{3.16}$$

$$\int_0^t L(\xi)\sin\left(Q^{\frac{1}{2}}(\xi,t-\tau)\right)\hat{g}(\xi,\tau)d\tau,\ t\in(0,T).$$

By using (3.2), (3.16) in a similar way, we have

$$\|L_1*u_t\|_{X_\infty}\leq C\left[\|\varphi\|_{Y^{s,p}}+\|\varphi\|_{X_1}+\right. \tag{3.17}$$

$$\|\psi\|_{Y^{s,p}}+\|\psi\|_{X_1}+\int_0^t\left(\|g(.,\tau)\|_{Y^{s,p}}+\|g(.,\tau)\|_{X_1}\right)d\tau\Bigg].$$

Then from (3.15) and (3.17), we obtain the assertion. Let now, we show that problem (3.1) has a unique solution $u\in C^{(1)}([0,T];Y^{s,p})$. Let's admit it is the opposite. So let's assume that the problem (3.1) has two solutions u_1, $u_2\in C^{(1)}([0,T];Y^{s,p})$. Then by linearity of (3.1), we get that $\upsilon=u_1-u_2$ is also a solution of the corresponding homogenous equation

$$L_0*\upsilon_{tt}+L_1*\upsilon=0,\ \upsilon(x,0)=0,\ \upsilon_t(x,0)=0,\ x\in\mathbb{R}^n,\ t\in(0,T).$$

Moreover, by estimate (3.8) we have the following estimate

$$\|L_1*\upsilon\|_{X_\infty}\leq 0. \tag{3.18}$$

Here, (3.18) implies that $L_1*\upsilon=0$. By virtue of Remark 3.1, the null space of the operator L_1 equal $\{0\}$. Hence, we get $\upsilon=0$, i.e. $u_1=u_2$. □

By reasoning as in Theorem 3.1 we obtain

Theorem 3.2. *Assume the Condition 3.1 is satisfied. Then for* $\varphi,\ \psi,\ g(.,t)\in Y^{s,p}$, $t\in(0,T)$ *and* $g(x,.)\in L_1(0,T)$, $x\in\mathbb{R}^n$ *there exists a unique solution of the problem* (3.1) *satisfies the following the uniform estimate*

$$\|L_1*u\|_{Y^{s,p}}+\|L_1*u_t\|_{Y^{s,p}}\leq \tag{3.19}$$

$$C\left[\|\varphi\|_{Y^{s,p}}+\|\psi\|_{Y^{s,p}}+\int_0^t\|g(.,\tau)\|_{Y^{s,p}}d\tau\right].$$

Proof. From (3.4) we have the following uniform estimate

$$\left\| F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u} \right\|_{X_p} + \left\| F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}_t \right\|_{X_p} \leq \tag{3.20}$$

$$C\left\{ \left\| F^{-1}(1+|\xi|)^{\frac{s}{2}} C(\xi,t)\,\hat{\varphi} \right\|_{X_p} + \left\| F^{-1}(1+|\xi|)^{\frac{s}{2}} S(\xi,t)\,\hat{\psi} \right\|_{X_p} + \right.$$

$$\left. \int_0^t \left\| (1+|\xi|)^{\frac{s}{2}}\, \hat{\Phi}(\xi,t-\tau)\,\hat{g}(.,\tau) \right\|_{X_p} d\tau \right\}.$$

By Condition 3.1 and by virtue of Fourier multiplier theorems (see e.g. [28, Theorem 4.3]), we get that $\left(1+|\xi|^2\right)^{-\frac{s}{2}} C(\xi,t)$, $\left(1+|\xi|^2\right)^{-\frac{s}{2}} S(\xi,t)$ and $\left(1+|\xi|^2\right)^{-\frac{s}{2}} \hat{\Phi}(\xi,t)$ are Fourier multipliers in $L_p(\mathbb{R}^n)$ uniformly with respect to $t \in [0,T]$. So, the estimate (3.20) by using the Minkowski's inequality for integrals implies (3.19).

The uniqueness of solution is obtained by reasoning as in Theorem 3.1. □

4. Initial value problem for nonlinear equation

In this section, we will show the local existence and uniqueness of solution for the Cauchy problem (1.1)–(1.2).

For the study of the nonlinear problem (1.1)–(1.2) we need the following lemmas

Lemma 4.1 *(Nirenberg's inequality). [29]. Assume that* $u \in L_p(\Omega)$, $D^m u \in L_q(\Omega)$, $p,q \in (1,\infty)$. *Then for* i *with* $0 \leq i \leq m$, $m > \frac{n}{q}$ *we have*

$$\left\| D^i u \right\|_r \leq C \|u\|_p^{1-\mu} \sum_{k=1}^{n} \left\| D_k^m u \right\|_q^{\mu}, \tag{4.1}$$

where

$$\frac{1}{r} = \frac{i}{m} + \mu\left(\frac{1}{q} - \frac{m}{n}\right) + (1-\mu)\frac{1}{p},\ \frac{i}{m} \leq \mu \leq 1.$$

Lemma 4.2. *[30]. Assume that* $u \in W_p^m(\Omega) \cap L_\infty(\Omega)$ *and* $f(u)$ *possesses continuous derivatives up to order* $m \geq 1$. *Then* $f(u) - f(0) \in W_p^m(\Omega)$ *and*

$$\|f(u) - f(0)\|_p \leq \left\| f^{(1)}(u) \right\|_\infty \|u\|_p,$$

$$\left\| D^k f(u) \right\|_p \leq C_0 \sum_{j=1}^{k} \left\| f^{(j)}(u) \right\|_\infty \|u\|_\infty^{j-1} \left\| D^k u \right\|_p,\ 1 \leq k \leq m, \tag{4.2}$$

where $C_0 \geq 1$ *is a constant.*

Let

$$E_0=\left(Y^{s,p},X_p\right)_{\frac{1}{2p},p}=B_p^{s\left(1-\frac{1}{p}\right)}\left(\mathbb{R}^n\right).$$

Remark 4.1. By using J. Lions-I. Petree result (see e.g. [31, § 1.8]) or [32] we obtain that the map $u\rightarrow u\left(t_0\right)$, $t_0\in[0,T]$ is continuous and surjective from $W_p^{2,s}\left(0,T;X_p,Y^{s,p}\right)$ onto E_0 and there is a constant C_1 such that

$$\left\|u\left(t_0\right)\right\|_{E_0}\leq C_1\left\|u\right\|_{Y^{s,p}},\ 1\leq p\leq\infty.$$

Let

$$C^{(i)}\left(s,p\right)=C^{(i)}\left([0,T];Y_{\infty}^{s,p}\right).$$

First all of, we define the space $Y\left(T\right)$ equipped with the norm defined by

$$\left\|u\right\|_{Y(T)}=\max_{t\in[0,T]}\left\|u\right\|_{Y^{s,p}}+\max_{t\in[0,T]}\left\|u\right\|_{X_{\infty}},\ u\in Y\left(T\right).$$

It is easy to see that $Y\left(T\right)$ is a Banach space. For φ, $\psi\in Y^{s,p}$, let

$$M=\left\|\varphi\right\|_{Y^{s,p}}+\left\|\varphi\right\|_{X_{\infty}}+\left\|\psi\right\|_{Y^{s,p}}+\left\|\psi\right\|_{X_{\infty}}.$$

Definition 4.1. For any $T>0$ if υ, $\psi\in Y_1^{s,p}$ and $u\in C^{(0)}\left(s,p\right)$ satisfies the problem (1.1)–(1.2) then $u\left(x,t\right)$ is called the continuous solution or the strong solution of (1.1)–(1.2). If $T<\infty$, then $u\left(x,t\right)$ is called the local strong solution of the problem (1.1)–(1.2). If $T=\infty$, then $u\left(x,t\right)$ is called the global strong solution of (1.1)–(1.2).

Condition 4.1. Assume:

(1) The Condition 3.1 holds;

(2) φ, $\psi\in Y_1^{s,p}$ for $1\leq p\leq\infty$;

(3) the function $u\rightarrow\hat{f}\left(\xi,t,u\right)$: $\mathbb{R}^n\times[0,T]\times E_0\rightarrow\mathbb{C}$ is a measurable in $\left(\xi,t\right)\in\mathbb{R}_T^n$ for $u\in E_0$ moreover, $\hat{f}\left(\xi,t,u\right)$ is continuous in $u\in E_0$ and $\hat{f}\left(\xi,t,u\right)\in C^{([s]+1)}\left(E_0;\mathbb{C}\right)$ uniformly for $\xi\in\mathbb{R}^n$ and $t\in[0,T]$.

Main aim of this section is to prove the following result:

Theorem 4.1. *Let the Condition 4.1 hold. Then problem* (1.1)–(3.2) *has a unique local strong solution* $u\in C^{(2)}\left(s,p\right)$, *where* T_0 *is a maximal time that is appropriately small relative to* M. *Moreover, if*

$$\sup_{t\in[0,T_0)}\left(\left\|u\right\|_{Y^{s,p}}+\left\|u\right\|_{X_{\infty}}+\left\|u_t\right\|_{Y^{s,p}}+\left\|u_t\right\|_{X_{\infty}}\right)<\infty \tag{4.3}$$

then $T_0=\infty$.

Proof. First, we are going to prove the existence and the uniqueness of the local continuous solution of the problem (1.1)–(1.2) by contraction mapping principle. From Lemma 4.2 we know that $f(u) \in L_p\left(0, T; Y_\infty^{s,p}\right)$ for any $T > 0$. Consider a map G on $Y(T)$ such that $G(u)$ is the operator defined by

$$G(u) = G(u)(x,t) = S_1(t)\varphi(x) + S_2(t)\psi(x) + O(u), \tag{4.4}$$

where

$$O(u) = \int_0^t F^{-1}\left[S(\xi, t-\tau) L(\xi) \hat{f}(u)(\xi,\tau)\right] d\tau,\ t \in (0,T), \tag{4.5}$$

where $S_1(t)$, $S_2(t)$ are linear operators defined by (3.6) and F^{-1} is the inverse Fourier transformation. From Lemma 4.2 it is easy to see that the map G is well defined for $f \in C^{(2)}(X_0; \mathbb{C})$. Let

$$W(u) = F^{-1}\left[S(\xi, t-\tau) L(\xi) f(u)\right](x,\tau).$$

By assumption (2) of Condition 4.1 and by virtue [28, Theorem 4.3], the function $U(\xi, t-\tau)L(\xi)$ is a Fourier multiplier theorem in X_p, i.e. if $f(u) \in X_p$, then $W(u) \in X_p$.

We put

$$Q(M;T) = \left\{u \mid u \in Y(T),\ \|u\|_{Y(T)} \leq M+1\right\}.$$

First, by reasoning as in [9] let us prove that the map G has a unique fixed point in $Q(M;T)$. For this aim, it is sufficient to show that the operator G maps $Q(M;T)$ into $Q(M;T)$ and G: $Q(M;T) \to Q(M;T)$ is strictly contractive if T is appropriately small relative to M. Consider the function $\bar{W}(\xi)$: $[0,\infty) \to [0,\infty)$ defined by

$$\bar{W}(\sigma) = \max_{|\xi| \leq \sigma}\left\{\left|\bar{W}^{(1)}(\xi)\right|, \left|\bar{W}^{(2)}(\xi)\right|, ..., \left|\bar{W}^{([s])}(\xi)\right|\right\}, \sigma \geq 0.$$

It is clear to see that the function $\bar{W}(\sigma)$ is continuous and nondecreasing on $[0,\infty)$. From Lemma 4.2 we have

$$\begin{gathered}\|W(u)\|_{Y^{2,p}} \leq \left\|W^{(1)}(u)\right\|_{X_\infty}\|u\|_{X_p} + \left\|W^{(1)}(u)\right\|_{X_\infty}\|Du\|_{X_p} + \\ C_0\left[\left\|W^{(1)}(u)\right\|_{X_\infty}\|u\|_{X_p} + ... + \left\|W^{([s])}(u)\right\|_{X_\infty}\|u\|_{X_\infty}\left\|D^{[s]}u\right\|_{X_p}\right] \leq \\ 2C_0\bar{W}(M+1)(M+1)\|u\|_{Y^{2,p}}.\end{gathered} \tag{4.6}$$

By using the Theorem 3.1 we obtain from (4.5):

$$\|G(u)\|_{X_\infty} \leq \|\varphi\|_{X_\infty} + \|\psi\|_{X_\infty} + \int_0^t \|W(x,\tau,u(\tau))\|_{X_\infty}, \tag{4.7}$$

$$\|G(u)\|_{Y^{2sp}} \leq \|\varphi\|_{Y^{s,p}} + \|\psi\|_{Y^{s,p}} + \int_0^t \|W(x,\tau,u(\tau))\|_{Y^{2,p}}\, d\tau. \tag{4.8}$$

Thus, from (4.6)–(4.8) and Lemma 4.2 we get

$$\|G(u)\|_{Y(T)} \leq M + T(M+1)\left[1 + 2C_0(M+1)\bar{f}(M+1)\right].$$

If T satisfies

$$T \leq \left\{(M+1)\left[1 + 2C_0(M+1)\bar{f}(M+1)\right]\right\}^{-1}, \tag{4.9}$$

then

$$\|Gu\|_{Y(T)} \leq M + 1.$$

Therefore, if (4.9) holds, then G maps $Q(M;T)$ into $Q(M;T)$. Now, we are going to prove that the map G is strictly contractive. Assume $T > 0$ and $u_1, u_2 \in Q(M;T)$ given. We get

$$G(u_1) - G(u_2) = \int_0^t \left[W(u_1)(x,\tau) - W(u_2)(x,\tau)\right] d\tau,\ t \in (0,T).$$

By using the assumption (3) and the mean value theorem, we obtain

$$\begin{gathered} W(u_1) - W(u_2) = W^{(1)}(u_2 + \eta_1(u_1 - u_2))(u_1 - u_2), \\ D[W(u_1) - W(u_2)] = W^{(2)}(u_2 + \eta_2(u_1 - u_2))(u_1 - u_2) D_\xi u_1 + \\ W^{(1)}(u_2)\left(Du_1 - D_\xi u_2\right), \\ D^2\left[\hat{f}(u_1) - \hat{f}(u_2)\right] = W^{(3)}(u_2 + \eta_3(u_1 - u_2))(u_1 - u_2)(Du_1)^2 + \\ W^{(2)}(u_2)(Du_1 - Du_2)(Du_1 + Du_2) + \\ W^{(2)}(u_2 + \eta_4(u_1 - u_2))(u_1 - u_2) D^2 u_1 + W^{(1)}(u_2)\left(D^2 u_1 - D^2 u_2\right), \end{gathered}$$

where $0 < \eta_i < 1$. Thus, using Holder's and Nirenberg's inequality, we have

$$\|W(u_1) - W(u_2)\|_{X_\infty} \leq \bar{W}(M+1)\|u_1 - u_2\|_{X_\infty}, \tag{4.10}$$

$$\|(u_1) - W(u_2)\|_{X_p} \leq \bar{W}(M+1)\|u_1 - u_2\|_{X_p},$$

$$\begin{gathered} \|D[W(u_1) - W(u_2)]\|_{X_p} \leq (M+1)\bar{W}(M+1)\|u_1 - u_2\|_{X_\infty} + \\ \bar{W}(M+1)\|W(u_1) - W(u_2)\|_{X_p}, \end{gathered} \tag{4.11}$$

$$\begin{gathered} \left\|D^2[W(u_1) - W(u_2)]\right\|_{X_p} \leq (M+1)\bar{W}(M+1)\|u_1 - u_2\|_{X_\infty}\left\|D^2 u_1\right\|^2_{Y^{2,p}} + \\ \bar{W}(M+1)\|D(u_1 - u_2)\|_{Y^{2,p}}\|D(u_1 + u_2)\|_{Y^{2,p}} + \end{gathered}$$

$$\bar{W}(M+1)\|u_1-u_2\|_{X_\infty}\left\|D^2u_1\right\|_{X_p}+\bar{W}(M+1)\|D(u_1-u_2)\|_{X_p}\leq$$
$$C^2\bar{W}(M+1)\|u_1-u_2\|_{X_\infty}\|u_1\|_{X_\infty}\left\|D^2u_1\right\|_{X_p}+ \tag{4.12}$$
$$C^2\bar{W}(M+1)\|u_1-u_2\|_{X_\infty}\left\|D^2(u_1-u_2)\right\|_{X_p}\|u_1+u_2\|_{X_\infty}\left\|D^2(u_1+u_2)\right\|_{X_p}$$
$$+(M+1)\bar{W}(M+1)\|u_1-u_2\|_{X_\infty}+\bar{W}(M+1)\left\|D^2(u_1-u_2)\right\|_{X_p}\leq$$
$$3C^2(M+1)^2\bar{W}(M+1)\|u_1-u_2\|_{X_\infty}+$$
$$2C^2(M+1)\bar{W}(M+1)\left\|D^2(u_1-u_2)\right\|_{X_p}.$$

In a similar way, we have

$$\left\|D^{[s]}[W(u_1)-W(u_2)]\right\|_{X_p}\leq \tag{4.13}$$
$$C_1\|u_1-u_2\|_{X_\infty}+C_2\left\|D^{[s]}(u_1-u_2)\right\|_{X_p}.$$

From (4.10)–(4.13), using Minkowski's inequality for integrals, Fourier multiplier theorems in X_p spaces and Young's inequality, we obtain

$$\|G(u_1)-G(u_2)\|_{Y(T)}\leq\int_0^t\|u_1-u_2\|_{X_\infty}d\tau+\int_0^t\|u_1-u_2\|_{Y^{s,p}}d\tau+$$
$$\int_0^t\|W(u_1)-W(u_2)\|_{X_\infty}d\tau+\int_0^t\|W(u_1)-W(u_2)\|_{Y^{s,p}}d\tau\leq$$
$$T\left[1+C_1(M+1)^2\bar{W}(M+1)\right]\|u_1-u_2\|_{Y(T)},$$

where C_1 is a constant. If T satisfies (4.9) and the following inequality

$$T\leq\frac{1}{2}\left[1+C_1(M+1)^2\bar{W}(M+1)\right]^{-1}, \tag{4.14}$$

then

$$\|Gu_1-Gu_2\|_{Y(T)}\leq\frac{1}{2}\|u_1-u_2\|_{Y(T)}.$$

That is, G is a contractive map. By contraction mapping principle we know that $G(u)$ has a fixed point $u(x,t)\in Q(M;T)$ that is a solution of (1.1)–(1.2). From (3.5) we get that u is a solution of the following integral equation

$$u(t,x)=S_1(t)\varphi(x)+S_2(t)\psi(x)++\int_0^t W(u)(x,\tau)d\tau,\ t\in(0,T).$$

Let us show that this solution is a unique in $Y(T)$. Let $u_1, u_2 \in Y(T)$ are two solutions of the problem (1.1)–(1.2). Then

$$u_1 - u_2 = \int_0^t \left[W(u_1)(x,\tau) - W(u_2)(x,\tau)\right] d\tau. \tag{4.15}$$

By definition of the space $Y(T)$, we can assume that

$$\|u_1\|_{X_\infty} \leq C_1(T), \ \ \|u_1\|_{X_\infty} \leq C_1(T).$$

Hence, by Minkowski's inequality for integrals and Theorem 3.2 we obtain from (4.15)

$$\|u_1 - u_2\|_{Y^{s,p}} \leq C_2(T) \int_0^t \|u_1 - u_2\|_{Y^{s,p}} d\tau. \tag{4.16}$$

From (4.16) and Gronwall's inequality, we have $\|u_1 - u_2\|_{Y^{s,p}} = 0$, i.e. problem (1.1)–(1.2) has a unique solution which belongs to $Y(T)$. That is, we obtain the first part of the assertion. Now, let $[0, T_0)$ be the maximal time interval of existence for $u \in Y(T_0)$. It remains only to show that if (4.3) is satisfied, then $T_0 = \infty$. Assume contrary that, (4.3) holds and $T_0 < \infty$. For $T \in [0, T_0)$, we consider the following integral equation

$$\upsilon(x,t) = S_1(t)\, u(x,T) + S_2(t)\, u_t(x,T) + \int_0^t W(\upsilon)(x,\tau)\, d\tau, \ t \in (0,T). \tag{4.17}$$

By virtue of (4.3), for $T' > T$ we have

$$\sup_{t\in[0,T)} \left(\|u\|_{Y^{s,p}} + \|u\|_{X_\infty} + \|u_t\|_{Y^{s,p}} + \|u_t\|_{X_\infty}\right) < \infty.$$

By reasoning as a first part of theorem and by contraction mapping principle, there is a $T^* \in (0, T_0)$ such that for each $T \in [0, T_0)$ the equation (4.17) has a unique solution $\upsilon \in Y(T^*)$. The estimates (4.9) and (4.14) imply that T^* can be selected independently of $T \in [0, T_0)$. Set $T = T_0 - \frac{T^*}{2}$ and define

$$\tilde{u}(x,t) = \begin{cases} u(x,t), \ t \in [0,T] \\ \upsilon(x, t-T), t \in \left[T, T_0 + \frac{T^*}{2}\right] \end{cases}. \quad \square \tag{4.18}$$

By construction $\tilde{u}(x,t)$ is a solution of the problem (1.1)–(1.2) on $\left[T, T_0 + \frac{T^*}{2}\right]$ and in view of local uniqueness, $\tilde{u}(x,t)$ extends u. This is against to the maximality of $[0, T_0)$, i.e. we obtain $T_0 = \infty$.

The solution in Theorem 4.1 can be extended to a maximal interval $[0, T_{\max})$, where finite $T_{\max}$ is characterized by the blow-up condition

$$\limsup_{T\to T_{\max}} \|u\|_{Y^{s,p}(A;E)} = \infty.$$

Lemma 4.3. *Let the problem (1.1)–(1.2) has a local strong solution $u \in C\left([0\ ,\ T_0)\,; Y_\infty^{s,p}\right)$, where $[0\ ,\ T_0)$ is a maximal time interval. Moreover, if*

$$\sup_{t\in[0,T]} \left(\|u\|_{Y_\infty^{s,p}} + \|u_t\|_{Y_\infty^{s,p}}\right) < \infty \tag{4.19}$$

then $T_0 = \infty$.

Proof. Indeed, by reasoning as in the second part of the proof of Theorem 4.1, by using a continuation of local solution of (1.1)–(1.2) and assuming contrary that, (4.19) holds and $T_0 < \infty$ we obtain contradiction, i.e. we get $T_0 = T_{\max} = \infty$. □

5. Conservation of energy and global existence

In this section we will study the regularity and global existence properties of the problem (1.1)–(1.2) in $L_2(\mathbb{R}^n)$. Here, scalar product $u, \upsilon \in L_2(\mathbb{R}^n)$ will denote just by (u, υ). Moreover, norm of $u \in L_2(\mathbb{R}^n)$ denotes by $\|u\|$.

Condition 5.1. Assume that the Condition 4.1 is satisfied for $p = 2$. Let $L_2(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$. Moreover, let

$$0 < L_2(\xi)\,|L_0(\xi)|^{-2} \leq C_L \left(1 + |\xi|^2\right)^{\frac{r}{2}}. \tag{5.0}$$

Consider the Fourier multiplier operator $S = S_g$ defined by

$$u \in D(S) = Y^{s,p},\ Vu = \mathbb{F}^{-1}\left[L_2^{-\frac{1}{2}}(\xi)\,\hat{u}(\xi)\right]. \tag{5.1}$$

Lemma 5.0. *Let the Condition 5.1 hold. Then the operator V is a self-adjoint operator in $L_2(\mathbb{R}^n)$ and the following are hold*

$$V^{-1}u = F^{-1}\left[L_2^{\frac{1}{2}}(\xi)\,\hat{u}(\xi)\right],\ V^{-2}u = -\Delta L_2 * u. \tag{5.2}$$

Proof. Indeed, for all $u, \upsilon \in L_2(\mathbb{R}^n)$ by Parseval's identity we have

$$(Vu, \upsilon) = \left(F^{-1}\left[L_2^{-\frac{1}{2}}(\xi)\,\hat{u}(\xi)\right], \upsilon\right) = \int\limits_{\mathbb{R}^n} F^{-1}\left[L_2^{-\frac{1}{2}}(\xi)\,\hat{u}(\xi)\right]\upsilon(x)\,dx =$$

$$\int\limits_{\mathbb{R}^n}\left[L_2^{-\frac{1}{2}}(\xi)\,\hat{u}(\xi)\right]\hat{\upsilon}(\xi)\,d\xi = \int\limits_{\mathbb{R}^n} u(x)\,F^{-1}\left[L_2^{-\frac{1}{2}}(\xi)\,\hat{\upsilon}(\xi)\right]dx = (u, V\upsilon).$$

Moreover,

$$V^{-1}(Vu) = F^{-1}\left[L_2^{\frac{1}{2}}(\xi)\,F(Vu)\right] = F^{-1}\left[L_2^{\frac{1}{2}}(\xi)\,L_2^{-\frac{1}{2}}(\xi)\,\hat{u}(\xi)\right] =$$

$$F^{-1}\hat{u}(\xi) = u,\ V\left(V^{-1}u\right) = F^{-1}\left[L_2^{-\frac{1}{2}}(\xi)\,F\left(V^{-1}u\right)\right] =$$

$$F^{-1}\left[L_2^{-\frac{1}{2}}(\xi)\,L_2^{\frac{1}{2}}(\xi)\,\hat{u}(\xi)\right] = F^{-1}\hat{u}(\xi) = u.$$

That is the operator V has an inverse V^{-1} defined by (5.0). Moreover, we have

$$V^2 u = V(Vu) = F^{-1}\left[L_2^{-\frac{1}{2}}(\xi)\,F(L_2 u)\right] =$$

$$F^{-1}\left[L_2^{-\frac{1}{2}}(\xi)\,L_2^{-\frac{1}{2}}(\xi)\,\hat{u}(\xi)\right] = F^{-1}\left[L_2^{-1}(\xi)\right].$$

Then

$$V^2[L_2 * u] = F^{-1}\left[L_2^{-1}(\xi)\,F(L_2 * u)\right] =$$

$$F^{-1}\left[L_2^{-1}(\xi)\,L_2(\xi)\,\hat{u}(\xi)\right] = F^{-1}\hat{u}(\xi) = u. \tag{5.3}$$

Here, (5.1) implies the second equality of (5.2). Hence, it is clear to see that

$$V^{-2}u = L_2 * u,\ V^{-1}u = \mathbb{F}^{-1}\left[L_2^{\frac{1}{2}}(\xi)\,\hat{u}(\xi)\right],$$

$$V^2 u = \mathbb{F}^{-1}\left[L_2^{-1}(\xi)\,\hat{u}(\xi)\right]. \tag{5.4}$$

Let

$$B(u) = V^2[L_1 * u - L_2 * f(u)] = \mathbb{F}^{-1}\left[L_1(\xi)\,L_2^{-1}(\xi)\,\hat{u}(\xi)\right] - f(u),$$

$$\Phi(\eta) = \int_0^{\eta} B(\sigma)\,d\sigma. \quad \square \tag{5.5}$$

We prove the following results

Lemma 5.1. *Let the Condition 5.1 hold,* $\varphi,\ \psi \in Y^{s,2} \cap X_\infty$ *and let* $u \in C^{(2)}\left(([0,T]; W^s)\right)$ *be solution of* (1.1)–(1.2) *for any* $t \in [0,T)$. *Then the energy*

$$E(t) = \|V(L_0 * u_t)\|^2 + 2\int_{\mathbb{R}^n} \Phi(u)\,L_0 * u_t\,dx, \tag{5.6}$$

is constant.

Proof. By Lemma 5.0 and by (5.3)-(5.5) it follows from straightforward calculation that

$$\frac{d}{dt}E\left(t\right)=2\left(V\left(L_0*u_{tt}\right),V\left(L_0*u_t\right)\right)+2\int\limits_{\mathbb{R}^n}\Phi_u\left(L_0*u_t\right)dx=$$

$$2\left(L_0*u_{tt},V^2L_0*u_t\right)+2\int\limits_{\mathbb{R}^n}V^2\left[L_1*u-L_2*f\left(u\right)\right]\left(L_0*u_t\right)dx=$$

$$2\left(L_0*u_{tt},V^2L_0*u_t\right)+2\left[\left(L_1*u-L_2*f\left(u\right),V^2L_0*u_t\right)\right]=$$

$$2\left[\left(L_0*u_{tt}+L_1*u-L_2*f\left(u\right),V^2L_0*u_t\right)\right]=0.$$

Hence, we obtain the assertion. □

By using the above lemmas we obtain the following results

Theorem 5.1. *Assume that the Condition 5.1 is satisfied and* φ, $\psi\in Y^{s,2}\cap X_\infty$. *Let* $s\geq 0$, $r>1$ *and there is some* $k>0$ *so that*

$$\Phi\left(\sigma\right)\geq -k\left|\sigma\right|^2, \text{for all } \sigma\in\mathbb{R}. \tag{5.7}$$

Then problem (1.1)–(1.2) *has a global solution*

$$u\in C^{(2)}\left(\left(\left(0,\infty\right);Y^{s,2}\right)\right).$$

Proof. By Theorem 4.1 we have the local existence of the solution $u\in C^{(2)}\left(\left([0,T];Y^{s,2}\right)\right)$ for some $T>0$. By assumption (5.7), we obtain

$$\left\|V\left(L_0*u_t\right)\right\|^2\leq E\left(0\right)+2k\left\|u\left(.,t\right)\right\|^2, \tag{5.8}$$

for all $t\in[0,T)$.

By virtue (5.1) and Plancherel equality we have

$$\left\|V\left(L_0*u_t\right)\right\|^2=\int\limits_{\mathbb{R}^n}L_2^{-1}\left(\xi\right)\left|L_0\left(\xi\right)\right|^2\left|\hat{u}_t\left(\xi\right)\right|^2d\xi. \tag{5.9}$$

From (5.0), (5.8) and (5.9) we get

$$\left\|V\left(L_0*u_t\right)\right\|^2\geq C_L^{-1}\int\limits_{\mathbb{R}^n}\left(1+\left|\xi\right|^2\right)^{\frac{r}{2}}\left|\hat{u}_t\left(\xi\right)\right|^2d\xi\geq C_L^{-1}\left\|u\right\|^2_{Y^{s,2}}. \tag{5.10}$$

By the triangle inequality, for any Banach space valued differentiable function υ we have

$$\frac{d}{dt}\left\|\upsilon\left(t\right)\right\|\leq\left\|\frac{d}{dt}\upsilon\left(t\right)\right\|.$$

Then in view of (5.8)–(5.10), we obtain

$$\frac{d}{dt}\left\|u\left(.,t\right)\right\|_{Y^{r,2}}^{2}\leq 2\left\|u_{t}\left(.,t\right)\right\|_{Y^{r,2}}\left\|u\left(.,t\right)\right\|_{Y^{r,2}}\leq$$
$$\left\|u_{t}\left(.,t\right)\right\|_{Y^{r,2}}^{2}+\left\|u\left(.,t\right)\right\|_{Y^{r,2}}^{2}\leq C_{L}E\left(0\right)+\left(2C_{L}k+1\right)\left\|u\left(t\right)\right\|_{Y^{r,2}}^{2}.$$

Gronwall's lemma implies that $\|u\left(.,t\right)\|_{Y^{r,2}}$ is bounded in $[0,T)$. Since $r>\frac{n}{2}$, we conclude that $\|u(t)\|_{X_\infty}$ also is bounded in $[0,T)$. By Lemma 4.3 this implies a global solution. □

6. Blow up in finite time

We will use the following lemma to prove blow up in finite time.

Lemma 6.1. *[33]. Suppose $H(t)$, $t\geq 0$ is a positive, twice differentiable function satisfying*

$$H^{(2)}H-(1+\mu)\left(H^{(1)}\right)^{2}\geq 0 \text{ for } \mu>0.$$

If $H(0)>0$ and $H^{(1)}(0)>0$, then $H(t)\to\infty$ when $t\to t_1$ for some

$$t_{1}\leq H(0)\left[\nu H^{(1)}(0)\right]^{-1}.$$

We rewrite the energy identity as

$$E\left(t\right)=\left\|V\left(L_{0}*u_{t}\right)\right\|^{2}+2\int\limits_{\mathbb{R}^{n}}\Phi\left(u\right)L_{0}*u_{t}dx, \tag{6.1}$$

where $\Phi\left(u\right)$ is defined by (5.5).

We prove here the following result

Theorem 6.1. *Assume the Condition 5.1 is satisfied. Let $u\in C^{(2)}\left(\left([0,T];Y^{s,2}\right)\right)$ be solution of (1.1)–(1.2) for any $t\in[0,T)$. Suppose there are a positive number ν such that*

$$\sigma B\left(\sigma\right)\geq -2\nu\Phi\left(\sigma\right) \text{ for } \nu\geq\mu \text{ and for all } \sigma\in\mathbb{R}. \tag{6.2}$$

Moreover,

$$E\left(0\right)=\left\|V\left(L_{0}*u_{t}\right)\right\|^{2}+2\int\limits_{\mathbb{R}^{n}}\Phi\left(u\right)L_{0}*u_{t}dx<0, \tag{6.3}$$

where $B\left(u\right)$ is defined by (5.5), i.e.

$$B\left(u\right)=\mathbb{F}^{-1}\left[L_{1}\left(\xi\right)L_{2}^{-1}\left(\xi\right)\hat{u}\left(\xi\right)\right]-f\left(u\right).$$

Then the solution u of the problem (1.1)–(1.2) blows up in finite time.

Proof. Assume that there is a global solution. Let

$$H(t) = \|Vw\|^2 + b(t+t_0)^2$$

for some positive b and t_0 that will be determined later, where

$$w = L_0 * u.$$

We have

$$\begin{gathered} H^{(1)}(t) = 2\left[(V(L_0 * u), V(L_0 * u_t)) + b(t+t_0)\right], \\ H^{(2)}(t) = 2\|V(L_0 * u_t)\|^2 + 2\Big((L_0 * u), V^2(L_0 * u_{tt})\Big) + 2b. \end{gathered} \tag{6.4}$$

Hence, from (1.1) by Parseval's equality, we get

$$\begin{gathered} \Big(L_0 * u, V^2(L_0 * u_{tt})\Big) = \Big(L_0 * u, V^2(L_2 * f(u))\Big) - \Big(u, V^2(L_1 * u)\Big) = \\ \Big(L_0 * u, \mathbb{F}^{-1}\Big[L_1(\xi) L_2^{-1}(\xi)\hat{u}(\xi)\Big] - f(u)\Big) = (L_0 * u, B(u)). \end{gathered} \tag{6.5}$$

From (6.2)–(6.3) and (6.5) we deduced

$$\begin{gathered} \Big(L_0 * u, V^2(L_0 * u_{tt})\Big) = \int\limits_{\mathbb{R}^n} B(u)(L_0 * u)\,dx \geq -2\nu \int\limits_{\mathbb{R}^n} \Phi(u) L_0 * u_t dx = \\ \nu\Big[\|V(L_0 * u_t)\|^2 - E(0)\Big]. \end{gathered} \tag{6.6}$$

From (6.4) and (6.6), we obtain

$$H^{(2)}(t) \geq (2+2\nu)\|Vw_t\|^2 - 2\nu[E(0)] + 2b. \tag{6.7}$$

On the other hand, in view of Cauchy-Schwartz inequality and by choosing $b(t+t_0) \geq 1$ we have

$$\begin{gathered} \Big(H^{(1)}(t)\Big)^2 = [2(Vw, Vw_t) + 2b(t+t_0)]^2 \leq \\ 4\Big[\|Vw\|^2\|Vw_t\|^2 + b(t+t_0)^2\Big(\|Vw\|^2 + \|Vw_t\|^2\Big)\Big] + 4b^2(t+t_0)^2. \end{gathered} \tag{6.8}$$

Hence, combining (6.4), (6.7) and (6.8) we obtain

$$\begin{gathered} H^{(2)}H - (1+\mu)\Big(H^{(1)}\Big)^2 \geq \\ \Big[(2+2\nu)\|Vw_t\|^2 - 2\nu E(0) + 2b\Big]\Big[\|Vw\|^2 + b(t+t_0)^2\Big] - \\ 4(1+\mu)\Big[\|Vw\|^2\|Vw_t\|^2 - b^2(t+t_0)^2\Big(\|Vw\|^2 + \|Vw_t\|^2 + 1\Big)\Big] = \end{gathered}$$

$$4(t+t_0)^2\left[(1+\nu)b^2-(1+\mu)\right]\|Vw_t\|^2+[4(\nu-\mu)]\|Vw_t\|^2\|Vw\|^2+$$
$$b(t+t_0)^2[2b-2\nu E(0)-4(1+\mu)b]+$$
$$\left[2b-2\nu E(0)-4(1+\mu)b^2(t+t_0)^2\right]\|Vw\|^2.$$

Hence, we choose b such that $(1+\nu)b^2-(1+\mu)\geq 0$, $2b-2\nu E(0)-4(1+\mu)b\geq 0$ and $2b-2\nu E(0)-4(1+\mu)b^2(t+t_0)^2\geq 0$, i.e.

$$b^2\geq\frac{1+\mu}{1+\nu},\ b\leq-\frac{\nu}{(1+2\mu)}E(0)$$
$$4(1+\mu)b^2(t+t_0)^2-2b\leq-2\nu E(0).$$

Then this gives

$$H^{(2)}H-(1+\mu)\left(H^{(1)}\right)^2\geq 0.$$

Moreover,

$$H^{(1)}(0)=2(\varphi,\psi)+2b(t_0)\geq 0$$

for sufficiently large t_0. According to Lemma 6.1, this implies that $H(t)$, and thus $\|u(t)\|^2$ blows up in finite time contradicting the assumption that the global solution exists. □

7. Applications

In this section we give some application of Theorem 4.1.

1. Let

$$L_0u=L_1u=L_2u=A_1u=\sum_{|\alpha|\leq 2}a_\alpha * D^\alpha u,$$

where a_α are complex numbers.

Then the problem (1.1)–(1.2) is reduced to the Cauchy problem for the following wave equation

$$A_1u_{tt}+A_1u=A_1f(x,t,u),\ x\in\mathbb{R}^2,\ t\in(0,T), \tag{7.1}$$
$$u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x).$$

Assume

$$A_1(\xi)=\sum_{k,j=1}^{2}a_{kj}\xi_k\xi_j>|\xi|^2\ \text{ for }\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$$

Hence, the Condition 3.1 is satisfied.

Let

$$X_p = L_p\left(\mathbb{R}^2\right), 1 \leq p \leq \infty, Y^{s,p} = L_p^s\left(\mathbb{R}^2\right).$$

Hence, from Theorem 4.1 we obtain:

Theorem 7.1. *Let* $s > n\left(1 + \frac{1}{q} + \frac{1}{p}\right)$ *with a* $p \in [1, \infty]$ *and a* $q \in [1, 2]$. *Assume that the function* $u \to f(x, t, u) : \mathbb{R}^2 \times [0, T] \times B_p^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^2\right) \to L_p\left(\mathbb{R}^2\right)$ *is measurable in* $(x, t) \in \mathbb{R}^2 \times [0, T]$ *for* $u \in B_p^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^2\right)$. *Moreover,* $f(x, t, u)$ *is continuous in* $u \in B_p^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^2\right)$ *and*

$$f(x, t, u) \in C^{(3)}\left(B_p^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^2\right); \mathbb{C}\right)$$

uniformly with respect to $(x, t) \in \mathbb{R}^2 \times [0, T]$. *Then for* φ, $\psi \in Y_1^{s,p}$ *problem* (7.1) *has a unique local strong solution* $u \in C^{(2)}\left([0, T_0); Y_\infty^{s,p}\right)$, *where* T_0 *is a maximal time interval that is appropriately small relative to* M. *Moreover, if*

$$\sup_{t\in[0,T_0}\left(\|u\|_{Y^{s,p}} + \|u\|_{X_\infty} + \|u_t\|_{Y^{s,p}} + \|u_t\|_{X_\infty}\right) < \infty$$

then $T_0 = \infty$.

2. Let

$$L_0 u = L_1 u = L_2 u = A_2 u = \sum_{|\alpha|\leq 4} a_\alpha * D^\alpha u,$$

where a_α are complex numbers, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, α_k are natural numbers and $|\alpha| = \sum_{k=1}^{3} \alpha_k$.

Then the problem (1.1)–(1.2) is reduced to the Cauchy problem for the following wave equation

$$A_2 u_{tt} + A_2 u = A_2 f(x, t, u), x \in \mathbb{R}^3, t \in (0, T), \tag{7.2}$$
$$u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x).$$

Assume

$$A_2(\xi) = \sum_{|\alpha|\leq 4} a_\alpha (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} (i\xi_3)^{\alpha_3} > |\xi|^2 \text{ for all } \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.$$

Therefore, the Condition 3.1 is satisfied.

Let

$$X_p = L_p\left(\mathbb{R}^3\right),\, 1 \leq p \leq \infty,\, Y^{s,p} = L_p^s\left(\mathbb{R}^3\right).$$

Hence, from Theorem 4.1 we obtain:

Theorem 7.2. *Let $s > n\left(1+\frac{1}{q}+\frac{1}{p}\right)$ with a $p \in [1,\infty]$ and a $q \in [1,2]$. Suppose that the function $u \to f(x,t,u)$: $\mathbb{R}^3 \times [0,T] \times B_p^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^3\right) \to L_p\left(\mathbb{R}^3\right)$ is measurable in $(x,t) \in \mathbb{R}^3 \times [0,T]$ for $u \in B_p^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^3\right)$. Moreover, $f(x,t,u)$ is continuous in $u \in B_p^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^3\right)$ and*

$$f(x,t,u) \in C^{(3)}\left(B_p^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^3\right);\mathbb{C}\right)$$

uniformly with respect to $(x,t) \in \mathbb{R}^3 \times [0,T]$. Then for φ, $\psi \in Y_1^{s,p}$ problem (7.2) has a unique local strong solution

$$u \in C^{(2)}\left([0,T_0);Y_\infty^{s,p}\right),$$

where T_0 is a maximal time interval that is appropriately small relative to M. Moreover, if

$$\sup_{t\in[0,T_0)}\left(\|u\|_{Y^{s,p}} + \|u\|_{X_\infty} + \|u_t\|_{Y^{s,p}} + \|u_t\|_{X_\infty}\right) < \infty$$

then $T_0 = \infty$.

3. Let

$$L_0 u = \sum_{|\alpha|\leq 4} a_{0\alpha} * D^\alpha,\, L_1 u = \sum_{|\alpha|\leq 2} a_{1\alpha} * D^\alpha,\, L_2 u = \sum_{|\alpha|\leq 4} a_{2\alpha} * D^\alpha,$$

where $a_{\alpha i}$ are complex numbers, $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, α_k are natural numbers and $|\alpha| = \sum_{k=1}^{3}\alpha_k$.

Then the problem (1.1)–(1.2) is reduced to Cauchy problem for the following wave equation

$$L_0 u_{tt} + L_1 u = L_2 f(x,t,u),\, x \in \mathbb{R}^3,\; t \in (0,T), \tag{7.3}$$
$$u(x,0) = \varphi(x),\, u_t(x,0) = \psi(x).$$

Assume

$$L_0(\xi) > 0 \text{ for } \xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3.$$

Since $m_0 - m_1 = m_2 - m_1 = 2$ the Condition 3.1 is satisfied.
Hence, from Theorem 4.1 we obtain:

Theorem 7.3. *Let* $s > n\left(1 + \frac{1}{q} + \frac{1}{p}\right)$ *with a* $p \in [1, \infty]$ *and a* $q \in [1, 2]$*. Assume the Condition 3.1 is satisfied. Suppose that the function* $u \to f(x, t, u)$*:* $\mathbb{R}^3 \times [0, T] \times B_p^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^3\right) \to L_p\left(\mathbb{R}^3\right)$ *is measurable in* $(x, t) \in \mathbb{R}^3 \times [0, T]$ *for* $u \in B_p^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^3\right)$*. Moreover,* $f(x, t, u)$ *is continuous in* $u \in B_p^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^3\right)$ *and*

$$f(x, t, u) \in C^{(3)}\left(B_p^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^3\right); \mathbb{C}\right)$$

uniformly with respect to $(x, t) \in \mathbb{R}^3 \times [0, T]$*. Then for* $\varphi, \psi \in Y_\infty^{s,p}$ *problem* (7.3) *has a unique local strong solution*

$$u \in C^{(2)}\left([0, T_0); Y_\infty^{s,p}\right),$$

where T_0 *is a maximal time interval that is appropriately small relative to* M*. Moreover, if*

$$\sup_{t \in [0, T_0)} \left(\|u\|_{Y^{s,p}} + \|u\|_{X_\infty} + \|u_t\|_{Y^{s,p}} + \|u_t\|_{X_\infty}\right) < \infty$$

then $T_0 = \infty$.

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