



A Comparison Between MMAE and SCEM for Solving Singularly Perturbed Linear Boundary Layer Problems

Süleyman Cengizci^a

^aComputer Programming, Antalya Bilim University, 07190, Antalya, Turkey

Abstract. In this study, we propose an efficient method so-called *Successive Complementary Expansion Method (SCEM)*, that is based on generalized asymptotic expansions, for approximating to the solutions of singularly perturbed two-point boundary value problems. In this easy-applicable method, in contrast to the well-known method *the Method of Matched Asymptotic Expansions (MMAE)*, any matching process is not required to obtain uniformly valid approximations. The key point: A uniformly valid approximation is adopted first, and complementary functions are obtained imposing the corresponding boundary conditions. An illustrative and two numerical experiments are provided to show the implementation and numerical properties of the present method. Furthermore, MMAE results are also obtained in order to compare the numerical robustnesses of the methods.

1. Introduction

Many phenomena in biology, chemistry, physics and engineering sciences are modelled and formulated by boundary value problems associated with different kinds of differential equations. In this manner, a model which is formulated by an equation or a system containing positive small parameter(s) is referred to as *perturbed model* and a model which does not keep the positive small parameter(s) is named as *reduced or unperturbed model* [1]. If the perturbed model contains the small parameter(s) as coefficient(s) to the highest order derivative term(s), then the problem is referred to as *singularly perturbed problem*, otherwise called as *regularly perturbed problem*.

First studies on perturbation problems were conducted by L. Prandtl and J. H. Poincaré in 1900's. The term "*boundary layer*" first appeared in Prandtl's paper "*Motion of fluids with very little viscosity*" in 1904. In this work, the small parameter was the inverse Reynold number and the equations were based on the classical Navier-Stokes equations of fluid mechanics [2]. Since 1900's, various methods have been constructed and employed. Kadalbajoo and Reddy [3] made a survey of various asymptotic and numerical methods that were developed between 1908 -1986 for the determination of approximate solutions of singular perturbation problems, then Kadalbajoo and Patidar [4] extended the work done by Kadalbajoo and Reddy and they surveyed the studies done by various researchers in the field of singular perturbation problems between 1984-2000 considering only one dimensional problems and made discussion on linear, nonlinear, semilinear and quasi-linear problems. Subsequently, Kadalbajoo and Gupta [5] presented a great survey on

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Email address: suleyman.cengizci@antalya.edu.tr (Süleyman Cengizci)

computational methods for different types of singular perturbation problems solved by different researchers between 2000-2009. Kumar studied general methods for solving singularly perturbed problems arising in engineering in his work [6]. Later in [7], Roos made a survey in singularly perturbed partial differential equations covering the years 2008-2012. Besides these great papers and surveys, there are also great reference books on applications and theory of singular perturbation problems. Some of them may be given as by Van Dyke[8], Holmes [9], Johnson [10], Lagerstrom [11], O'Malley [12], Eckhaus [13], Verhulst [14], Paulsen [15], Nayfeh [16], Murdock [17], Murray [18], and by Hinch [19].

In this paper, we employ the MMAE to approximate to the solutions of singular perturbation problems first, and later present an efficient asymptotic method which was introduced by Jean Cousteix and Jacques Mauss in [20, 22] as an alternative method to the MMAE: Successive Complementary Expansion Method (SCEM). The main principle of SCEM is built on the purpose of obtaining uniformly valid approximation to the singular perturbation problems without any matching procedure. The method is, in general, applicable to singular perturbation problems that we can approximate by MMAE. To this end, we can also compare the SCEM results with previously obtained ones by MMAE. The first step in SCEM starts with looking for an approximation for outer region. The approximation is generally in good quality for the outer region, but not for the inner region. The main idea is, using boundary conditions, to add a correction term that complements the approximation. The procedure can be iterated using new corrections for new terms to improve the accuracy of approximation. The most important advantage of SCEM is its ability of giving uniformly valid approximation without any matching procedure. Boundary conditions are enough to implement the method. Moreover, the boundary conditions are satisfied exactly, but not asymptotically.

2. Description of the method

In this work, we deal with singularly perturbed second order linear two-point boundary value problems in the following form

$$\varepsilon y''(x) + p(x)y'(x) + q(x)y(x) = r(x), \quad a \leq x \leq b; \quad a, b \in \mathbb{R}$$

with the boundary conditions $y(a) = \alpha \in \mathbb{R}$ and $y(b) = \beta \in \mathbb{R}$, where $0 < \varepsilon \ll 1$, $p(x), q(x)$ and $r(x)$ are sufficiently smooth functions. As $\varepsilon \rightarrow 0^+$, the order of the differential equation is reduced and the equation that we call *reduced equation*

$$p(x)y'(x) + q(x)y(x) = r(x), \quad a \leq x \leq b$$

is formed. One can observe that there are two boundary conditions in the original problem but only one of them can be imposed to the reduced equation. Moreover, as ε tends to 0, because of the reduction of the order, rapid changes occur in the solution. The region in which these rapid changes occur is named as *inner layer* or *boundary layer*, and the layer in which the solution exhibits mildly changes is named as *outer layer*. The sign of the coefficient function $p(x)$ determines the type of the layer(s). Over the interval $[a, b]$, if $p(x) > 0$ for all x , then a boundary layer occurs at the left-end of the interval, if $p(x) < 0$ for all x , then a boundary layer occurs at the right-end of the interval and if $p(x)$ changes sign in (a, b) , then interior layer(s) occurs at the zero(s) of $p(x)$.

2.1. Asymptotic Approximations

Given two functions $\phi(x, \varepsilon)$ and $\phi_a(x, \varepsilon)$, defined in a domain Ω , are referred to as *asymptotically identical to the order $\delta(\varepsilon)$* if their difference is asymptotically smaller than $\delta(\varepsilon)$,

$$\phi(x, \varepsilon) - \phi_a(x, \varepsilon) = o(\delta(\varepsilon)), \tag{2.1.1}$$

where $\delta(\varepsilon)$ is an order function and ε is a positive small parameter arising from the physical problem under consideration. The function $\phi_a(x, \varepsilon)$ is named as an *asymptotic approximation* of the function $\phi(x, \varepsilon)$.

Asymptotic approximations, in general form, are defined by

$$\phi_a(x, \varepsilon) = \sum_{i=0}^n \delta_i(\varepsilon) \varphi_i(x, \varepsilon), \quad (2.1.2)$$

where the asymptotic sequence of order functions $\delta_i(\varepsilon)$ satisfies the condition $\delta_{i+1}(\varepsilon) = o(\delta_i(\varepsilon))$, as $\varepsilon \rightarrow 0$. In these conditions, the approximation (2.1.2) is named as *generalized asymptotic expansion*. If the expansion (2.1.2) is written in the form of

$$\phi_a(x, \varepsilon) = E_0 \phi = \sum_{i=0}^n \delta_i^{(0)}(\varepsilon) \varphi_i^{(0)}(x), \quad (2.1.3)$$

then it is named as *regular asymptotic expansion*. The special operator E_0 is called *outer expansion operator* at a given order $\delta(\varepsilon)$, thus $\phi - E_0 \phi = o(\delta(\varepsilon))$. For more detailed information on asymptotic approximations, we refer the interested reader to [9, 12, 13, 18, 20].

2.2. The MMAE for SCEM

Interesting cases occur when the function is not regular in Ω , so one of the approximations (2.1.2) and (2.1.3) is valid only in a restricted region $\Omega_0 \subset \Omega$, that is called the *outer region*. Here, in the simplest case, we introduce an inner domain which can be formally denoted as $\Omega_1 = \Omega - \Omega_0$ and located near the origin. In general, the boundary layer variable is expressed as $\bar{x} = \frac{x-x_0}{\xi(\varepsilon)}$ where x_0 is the point at which the rapid changes begin to occur and $\xi(\varepsilon)$ is the order of thickness of this boundary layer. If a regular expansion can be constructed in Ω_1 , we can write

$$\phi_a(x, \varepsilon) = E_1 \phi = \sum_{i=0}^n \delta_i^{(1)}(\varepsilon) \varphi_i^{(1)}(\bar{x}), \quad (2.2.1)$$

where the *inner expansion operator* E_1 is defined in Ω_1 of the same order $\delta(\varepsilon)$ just like the outer expansion operator E_0 ; thus, $\phi - E_1 \phi = o(\delta(\varepsilon))$. As a result,

$$\phi_a = E_0 \phi + E_1 \phi - E_1 E_0 \phi \quad (2.2.2)$$

is clearly uniformly valid approximation [11, 13, 21].

In MMAE, two distinct approximations are found for two distinct (outer and inner) regions and then to obtain a uniformly valid approximation over the whole domain, these approximations are matched using limit process. Despite all the valuable works devoted to MMAE, it is not possible to formulate a general mathematical theory of the method. Therefore, we will study it on an illustrative example.

Let us consider the following second order singularly perturbed problem that has been studied employing the MMAE in [9]

$$\varepsilon y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y(1) = 1, \quad 0 < x < 1. \quad (2.2.3)$$

This problem exhibits rapid changes near the point $x = 0$ as $\varepsilon \rightarrow 0^+$, this region is named as *boundary layer* or *inner layer*. But, over the other region which is called *outer region*, the solution does not show an unusual behavior. This region is the region which is far from the point $x = 0$. We will adopt an approximation for the outer solution as

$$y(x) \approx y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots \quad (2.2.4)$$

If approximation (2.2.4) is substituted into (2.2.3), we reach

$$\varepsilon (y_0''(x) + \varepsilon y_1''(x) + \dots) + 2(y_0'(x) + \varepsilon y_1'(x) + \dots) + 2(y_0(x) + \varepsilon y_1(x) + \dots) = 0, \quad (2.2.5)$$

and letting $\varepsilon = 0$, the reduced equation is obtained as

$$y_0' + y_0 = 0. \quad (2.2.6)$$

Solution of (2.2.6) is obviously $y_0(x) = Ae^{-x}$, $A \in \mathbb{R}$ and if the outer boundary condition is imposed (for $x = 1$), the outer solution is obtained as

$$y_0(x) = e^{1-x}. \quad (2.2.7)$$

In order to obtain inner (boundary) layer approximation, we introduce a new variable

$$\bar{x} = \frac{x}{\varepsilon}. \quad (2.2.8)$$

Thanks to the new variable, called *boundary layer (stretching) variable*, we get the chance to stretch the thin layer as $\varepsilon \rightarrow 0^+$. We denote the inner solution that depends on \bar{x} and valid for near the point $x = 0$ by $Y(\bar{x})$. Using the chain rule

$$\frac{d}{dx} = \frac{d\bar{x}}{dx} \frac{d}{d\bar{x}} = \frac{1}{\varepsilon} \frac{d}{d\bar{x}} \quad (2.2.9)$$

is obtained and applying the transformation (2.2.8) to the original problem (2.2.3)

$$\varepsilon^{-1} \frac{d^2 Y}{d\bar{x}^2} + 2\varepsilon^{-1} \frac{dY}{d\bar{x}} + 2Y = 0 \quad (2.2.10)$$

is found. If the both sides of (2.2.10) are multiplied by ε , we reach to

$$\frac{d^2 Y}{d\bar{x}^2} + 2 \frac{dY}{d\bar{x}} + 2\varepsilon Y = 0. \quad (2.2.11)$$

Equation (2.2.11) is a regularly perturbed linear ordinary differential equation and we propose an approximation in the form of

$$Y(\bar{x}) \approx Y_0(\bar{x}) + \varepsilon Y_1(\bar{x}) + \varepsilon^2 Y_2(\bar{x}) + \dots \quad (2.2.12)$$

Since we are interested only in the first term of the approximation (2.2.12), we can make our calculations for $\varepsilon = 0$. To this end, the equation

$$\frac{d^2 Y_0}{d\bar{x}^2} + 2 \frac{dY_0}{d\bar{x}} = 0 \quad (2.2.13)$$

is obtained. Considering the inner boundary condition at $x = 0$ ($x = 0 \implies \bar{x} = 0$), the general solution to the equation (2.2.13) is found as

$$Y_0(\bar{x}) = B(1 - e^{-2\bar{x}}), B \in \mathbb{R}. \quad (2.2.14)$$

It is obvious that we are not able to determine one of the unknown constants, B . To determine B , we will use the matching procedure of MMAE. So far we have found two approximations (2.2.7) and (2.2.14) that are valid for distinct regions, but we know that these two approximations actually belong to the same approximation. Using this idea, the matching procedure may be given as follows [9, 11]

$$\lim_{x \rightarrow 0^+} y_0(x) = \lim_{x \rightarrow 0^+} e^{1-x} = \lim_{\bar{x} \rightarrow \infty} B(1 - e^{-2\bar{x}}) = \lim_{\bar{x} \rightarrow \infty} Y_0(\bar{x}). \quad (2.2.15)$$

Thus, we reach

$$Y_0(\bar{x}) = (e - e^{1-2\bar{x}}). \quad (2.2.16)$$

Finally, we desire to obtain a composite approximation. To do this, we simply add the inner and outer approximations and subtract the common limit, that is, using the following procedure

$$y \approx y_0(x) + Y_0(\bar{x}) - Y_0(\infty) \\ \text{(or equivalently } y \approx y_0(x) + Y_0(\bar{x}) - y_0(0^+)) \quad (2.2.17)$$

we reach the following composite MMAE approximation

$$y_{\text{composite}} \approx e^{1-x} - e^{1-\frac{2x}{\varepsilon}}. \quad (2.2.18)$$

2.3. Successive Complementary Expansion Method

The uniformly valid SCEM approximation is in the regular form given as

$$y_n^{\text{scecm}}(x, \bar{x}, \varepsilon) = \sum_{i=0}^n \delta_i(\varepsilon) [y_i(x) + \Psi_i(\bar{x})] \quad (2.3.1)$$

where $\{\delta_i\}$ is an asymptotic sequence and functions $\Psi_i(\bar{x})$ are the complementary functions that depend on \bar{x} . Functions $y_i(x)$ are the outer approximation functions that have been found by MMAE before, and they only depend on x , not also on ε . If the functions $y_i(x)$ and $\Psi_i(\bar{x})$ also depend on ε , the uniformly valid SCEM approximation, that is named as *generalized SCEM approximation*, is given in the following form [22]-[24]

$$y_{ng}^{\text{scecm}}(x, \bar{x}, \varepsilon) = \sum_{i=0}^n \delta_i(\varepsilon) [y_i(x, \varepsilon) + \Psi_i(\bar{x}, \varepsilon)]. \quad (2.3.2)$$

Let us consider the problem (2.2.3) again and propose an approximation for $n = 0$, that is, we look for a SCEM approximation in the form of

$$y_0^{\text{scecm}}(x, \bar{x}, \varepsilon) = y_0(x, \varepsilon) + \Psi_0(\bar{x}, \varepsilon) \quad (2.3.3)$$

and we know that from the equation (2.2.7), $y_0(x) = e^{1-x}$. Thus, approximation (2.3.3) turns into

$$y_0^{\text{scecm}}(x, \bar{x}, \varepsilon) = e^{1-x} + \Psi_0(\bar{x}, \varepsilon). \quad (2.3.4)$$

Once the approximation (2.3.4) is substituted into the original problem (2.2.3)

$$\varepsilon \frac{d^2}{dx^2} y_0^{scem}(x, \bar{x}, \varepsilon) + 2 \frac{d}{dx} y_0^{scem}(x, \bar{x}, \varepsilon) + 2 y_0^{scem}(x, \bar{x}, \varepsilon) = 0 \tag{2.3.5}$$

is obtained. It follows that

$$\begin{aligned} \varepsilon \frac{d^2}{dx^2} (e^{1-x} + \Psi_0(\bar{x}, \varepsilon)) + 2 \frac{d}{dx} (e^{1-x} + \Psi_0(\bar{x}, \varepsilon)) + 2 (e^{1-x} + \Psi_0(\bar{x}, \varepsilon)) &= 0 \\ \varepsilon e^{1-\varepsilon \bar{x}} + \varepsilon \frac{1}{\varepsilon^2} \Psi_0''(\bar{x}, \varepsilon) - 2e^{1-\varepsilon \bar{x}} + \frac{2}{\varepsilon} \Psi_0'(\bar{x}, \varepsilon) + 2e^{1-\varepsilon \bar{x}} + 2\Psi_0(\bar{x}, \varepsilon) &= 0 \\ \frac{1}{\varepsilon} \Psi_0''(\bar{x}, \varepsilon) + \frac{2}{\varepsilon} \Psi_0'(\bar{x}, \varepsilon) + 2\Psi_0(\bar{x}, \varepsilon) &= -\varepsilon e^{1-\varepsilon \bar{x}} \\ \Psi_0''(\bar{x}, \varepsilon) + 2\Psi_0'(\bar{x}, \varepsilon) + 2\varepsilon \Psi_0(\bar{x}, \varepsilon) &= -\varepsilon^2 e^{1-\varepsilon \bar{x}}. \end{aligned} \tag{2.3.6}$$

The resulting equation that is given in the last line of (2.3.6) is a regularly perturbed linear non-homogeneous second order ordinary differential equation. In order to obtain first term of SCEM approximation, letting $\varepsilon = 0$, the first complementary SCEM approximation $\Psi_0(\bar{x}, \varepsilon)$ is obtained as the solution of the following two-point boundary value problem

$$\Psi_0''(\bar{x}, \varepsilon) + 2\Psi_0'(\bar{x}, \varepsilon) = 0, \quad \Psi_0(0, 0, \varepsilon) = -e, \quad \Psi_0\left(1, \frac{1}{\varepsilon}, \varepsilon\right) = 0.$$

The first SCEM approximation is found as

$$y_0^{scem}(x, \bar{x}, \varepsilon) = e^{1-x} + \frac{e(e^{-2\bar{x}} - 1)}{e^{\frac{-2}{\varepsilon}} - 1} - e.$$

On the other hand, Problem (2.2.3) is an exactly solvable problem and its exact solution is given as

$$y_{exact}(x) = \frac{(e^{\frac{x}{\varepsilon}(-1+\sqrt{1-2\varepsilon})} - e^{\frac{x}{\varepsilon}(-1-\sqrt{1-2\varepsilon})})}{e^{\frac{1}{\varepsilon}(-1+\sqrt{1-2\varepsilon})} - e^{\frac{1}{\varepsilon}(-1-\sqrt{1-2\varepsilon})}}. \tag{2.3.7}$$

ε	L^2 error in MMAE	L^2 error in SCEM
0.0001	0.000624687980610	0.000624687980610
0.0005	0.003124942983068	0.003124942983068
0.0010	0.006253648308213	0.006253648308213
0.0050	0.031287231692987	0.031287231692987
0.0100	0.061951928162705	0.061951928162705
0.0500	0.283794475853395	0.283794475853395
0.1000	0.498739448296190	0.498739403531008
0.3000	0.609196858399138	0.582659431139973
0.4000	0.540710242114810	0.416122748110290
0.6000	0.893536533505815	0.302123969698791
0.8000	1.754202335976711	0.240771410186466
1.0000	2.790418350467303	0.212524676097187

Table 1: L^2 -norm errors in MMAE and SCEM approximations for Illustrative Example

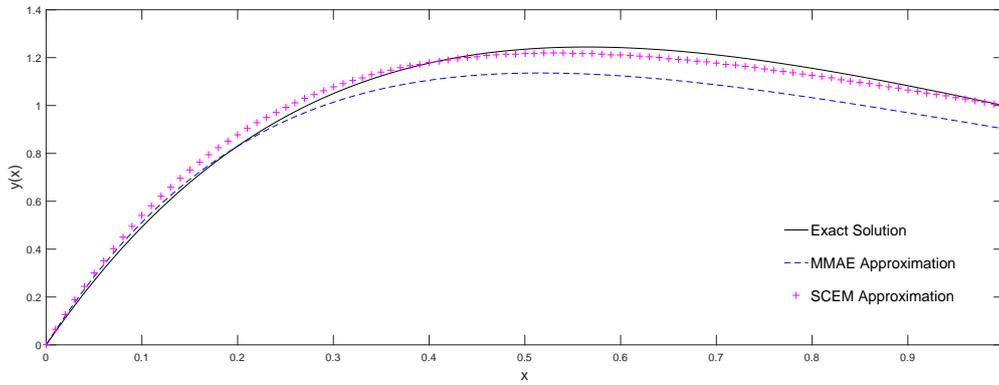


Figure 1: Comparison of results for illustrative example, $\varepsilon = 0.6$

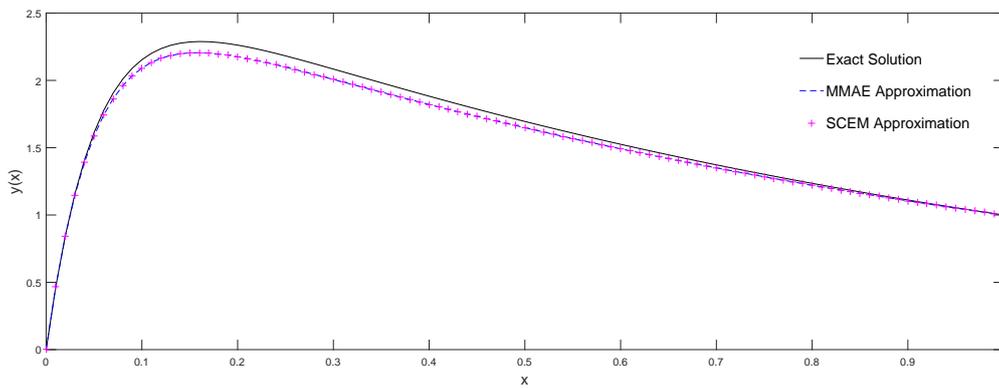


Figure 2: Comparison of results for illustrative example, $\varepsilon = 0.1$

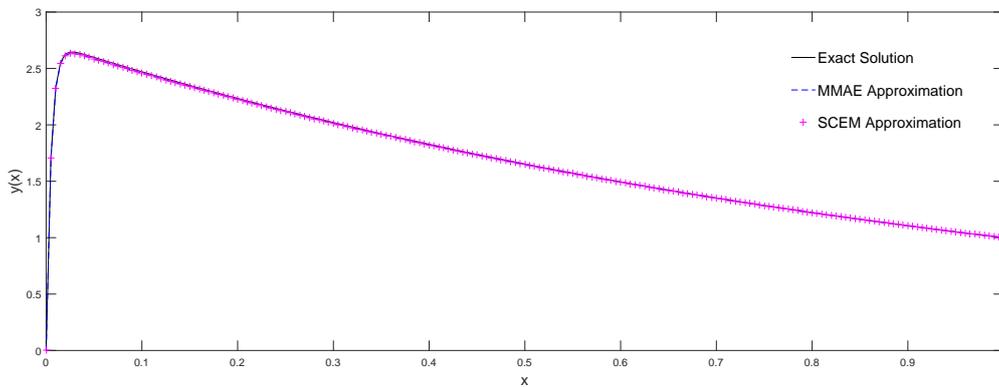


Figure 3: Comparison of results for illustrative example, $\varepsilon = 0.01$

In Fig.1 - Fig.3, exact solution, MMAE and SCEM approximations are given for the illustrative example.

Here, the SCEM approximation represents the expression given by $y_0^{sцем}(x, \bar{x}, \varepsilon) = e^{1-x} + \frac{e\left(e^{-\frac{2\bar{x}}{\varepsilon}} - 1\right)}{e^{-\frac{2}{\varepsilon}} - 1} - e$, that is obtained by balancing the equation (2.3.6) with respect to zeroth power (dominant order) of ε . While SCEM gives more accurate approximation for $\varepsilon = 0.6$, as ε gets smaller and smaller, SCEM and MMAE approximations overlap and approach to the exact solution. On the other hand, it can be observed especially in Fig.1 that MMAE approximations satisfy the boundary conditions asymptotically. Moreover, Table 1 shows that the SCEM approximations are in quite good agreement with exact solution even increasing ε values, and L^2 -norm errors for these two approximations are exactly same up to the values around $\varepsilon = 0.05$.

3. Numerical Experiments

In this section, two numerical examples are given. All the figures and numerical results are generated in Matlab 2016b environment.

Example 1: Consider the non-homogeneous singularly perturbed problem from fluid dynamics for fluid of small viscosity [25]

$$\varepsilon y'' + y' = 1 + 2x, \quad 0 \leq x < 1, \quad y(0) = 0, \quad y(1) = 1. \quad (3.1)$$

This problem has rapid changes near the point $x = 0$. The exact solution to (3.1) is given as

$$y_{exact}(x) = x^2 + (1 - 2\varepsilon)x + \frac{2\varepsilon - 1}{1 - e^{-\frac{1}{\varepsilon}}} \left(1 - e^{-\frac{x}{\varepsilon}}\right) \quad (3.2)$$

and one-term SCEM and MMAE approximations are obtained as follows.

The first term of the outer approximation, letting $\varepsilon = 0$, is obtained as the solution of initial value problem

$$y_0' = 1 + 2x, \quad y_0(1) = 1 \quad (3.3)$$

and it is clear that the solution is

$$y_0(x) = x^2 + x - 1. \quad (3.4)$$

Since the boundary layer occurs near the left-end of the interval, the stretching variable should be in the form of $\bar{x} = \frac{x}{\varepsilon}$. The first term of the inner approximation, letting $\varepsilon = 0$, is obtained as the solution of

$$Y_0'' + Y_0' = 0, \quad Y_0(\bar{x} = 0) = 0 \quad (3.5)$$

and it is clear that the solution is

$$Y_0(\bar{x}) = c_1(e^{-\bar{x}} - 1), \quad c_1 \in \mathbb{R}. \quad (3.6)$$

Now that the first terms of outer and inner approximations are obtained, these approximations should be matched by the definition of MMAE.

$$\lim_{x \rightarrow 0^+} y_0(x) = \lim_{x \rightarrow 0^+} (x^2 + x - 1) = \lim_{\bar{x} \rightarrow \infty} c_1(e^{-\bar{x}} - 1) = \lim_{\bar{x} \rightarrow \infty} Y_0(\bar{x}) \quad (3.7)$$

Thus, it is determined that

$$Y_0(\bar{x}) = e^{-\bar{x}} - 1. \quad (3.8)$$

ε	L^2 error in MMAE	L^2 error in SCEM
0.0001	0.001146036648629	0.001146036648629
0.0005	0.005730183242787	0.005730183242787
0.0010	0.011460350798498	0.011460350798498
0.0050	0.057047561195172	0.057047561195172
0.0100	0.112864481039688	0.112864481039688
0.0500	0.513159514418249	0.513159531588828
0.1000	0.901557814477664	0.901920266712351
0.3000	1.339243905573495	1.569217796713238
0.5000	1.057355422358385	1.719659382215715
0.6000	0.991379731889011	1.750119945541900
0.8000	1.119711839328917	1.782048657856496
1.0000	1.404221050193842	1.797421134420320

Table 2: L^2 -norm errors in MMAE and SCEM approximations for Example 1

Finally, the composite MMAE approximation to the problem (3.1) is found as

$$y_{composite}(x, \varepsilon) \approx x^2 + x - 1 + e^{-\frac{x}{\varepsilon}}. \tag{3.9}$$

Now, the one-term SCEM approximation can be determined substituting the approximation

$$y_0^{scem}(x, \bar{x}, \varepsilon) = y_0(x, \varepsilon) + \Psi_0(\bar{x}, \varepsilon) = x^2 + x - 1 + \Psi_0(\bar{x}, \varepsilon) \tag{3.10}$$

into the original problem (3.1):

$$\begin{aligned} \varepsilon \frac{d^2}{dx^2} (x^2 + x - 1 + \Psi_0(\bar{x}, \varepsilon)) + \frac{d}{dx} (x^2 + x - 1 + \Psi_0(\bar{x}, \varepsilon)) &= 1 + 2\varepsilon\bar{x} \\ \varepsilon \left(2 + \frac{\Psi_0''(\bar{x}, \varepsilon)}{\varepsilon^2} \right) + (2x + 1 + \frac{\Psi_0'(\bar{x}, \varepsilon)}{\varepsilon}) &= 1 + 2\varepsilon\bar{x} \\ 2\varepsilon^2 + \Psi_0''(\bar{x}, \varepsilon) + \varepsilon(2x + 1) + \Psi_0'(\bar{x}, \varepsilon) &= \varepsilon(1 + 2\varepsilon\bar{x}). \end{aligned} \tag{3.11}$$

In order to obtain first term of SCEM approximation, letting $\varepsilon = 0$, the first complementary SCEM approximation $\Psi_0(\bar{x}, \varepsilon)$ is obtained as the solution of the following two-point boundary value problem

$$\Psi_0''(\bar{x}, \varepsilon) + \Psi_0'(\bar{x}, \varepsilon) = 0, \Psi_0(0, 0, \varepsilon) = 1, \Psi_0(1, \frac{1}{\varepsilon}, \varepsilon) = 0. \tag{3.12}$$

Finally, the first SCEM and MMAE approximation are found as follows

$$y_0^{scem}(x, \bar{x}, \varepsilon) = x^2 + x - 1 + \frac{e^{-\frac{1}{\varepsilon}} - e^{-\frac{x}{\varepsilon}}}{e^{-\frac{1}{\varepsilon}} - 1}. \tag{3.13}$$

$$y_{composite}(x, \varepsilon) \approx x^2 + x - 1 + e^{-\frac{x}{\varepsilon}}. \tag{3.14}$$

Table 2 shows that MMAE approximations are slightly more accurate than SCEM approximations for non-homogeneous problem, Example 1. As one can point out from Fig.4 and Fig.5, as $\varepsilon \rightarrow 0^+$, MMAE and SCEM approximations are getting more accurate.

Example 2: Consider the singularly perturbed problem given in [26]

$$-\varepsilon y'' + y' + y = 1, 0 \leq x < 1, y(0) = 0, y(1) = 0. \tag{3.15}$$

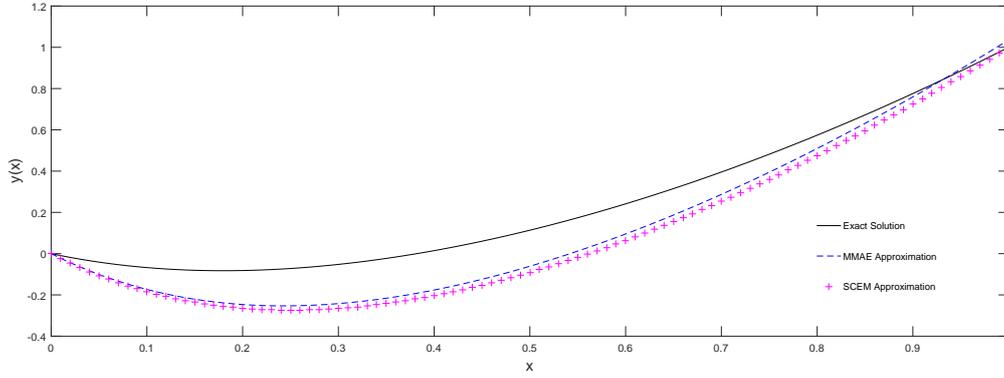


Figure 4: Comparison of results for Example 1, $\varepsilon = 0.3$

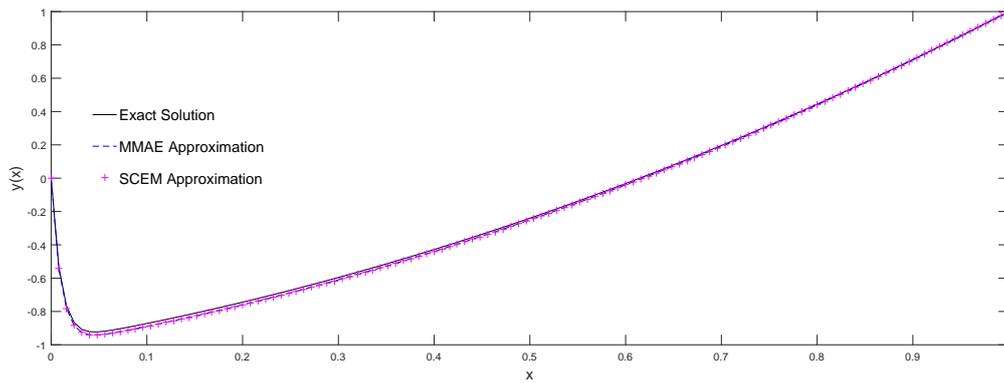


Figure 5: Comparison of results for Example 1, $\varepsilon = 0.01$

This problem has rapid changes near the point $x = 1$. The exact solution to (3.15) is given as

$$y(x) = 1 + \frac{e^{\lambda_1 x} (e^{\lambda_2} - 1)}{e^{\lambda_1} - e^{\lambda_2}} - \frac{e^{\lambda_2 x} (e^{\lambda_1} - 1)}{e^{\lambda_1} - e^{\lambda_2}}, \quad (3.16)$$

where $\lambda_1 = \frac{1 + \sqrt{1+4\varepsilon}}{2\varepsilon}$, $\lambda_2 = \frac{1 - \sqrt{1+4\varepsilon}}{2\varepsilon}$ and one-term SCEM and MMAE approximations are obtained as follows. The first term of the outer approximation is obtained letting $\varepsilon = 0$ in the original equation (3.15)

$$y' + y = 1, \quad y_0(0) = 0. \quad (3.17)$$

It is obvious that the solution is

$$y_0(x) = 1 - e^{-x}. \quad (3.18)$$

Since the problem (3.15) has boundary layer near the right end-point $x = 1$, the stretching variable should have a form $\bar{x} = \frac{x-1}{\varepsilon}$. The first term of the inner approximation, letting $\varepsilon = 0$, is obtained as the solution of

$$Y_0'' - Y_0' = 0, \quad Y_0(\bar{x} = 0) = 0 \quad (3.19)$$

and it is clear that the solution is

$$Y_0(\bar{x}) = c_1(e^{\bar{x}} - 1), \quad c_1 \in \mathbb{R}. \quad (3.20)$$

Now that the first terms of outer and inner approximations are obtained, these approximations should be matched by the definition of MMAE.

$$\lim_{x \rightarrow 1^-} y_0(x) = \lim_{x \rightarrow 1^-} (1 - e^{-x}) = \lim_{\bar{x} \rightarrow -\infty} c_1(e^{\bar{x}} - 1) = \lim_{\bar{x} \rightarrow -\infty} Y_0(\bar{x}) \quad (3.21)$$

Thus, it is determined that

$$Y_0(\bar{x}) = (e^{-1} - 1)(e^{\bar{x}} - 1). \quad (3.22)$$

Finally, the composite MMAE approximation to the problem (3.15) is found as

$$y_{\text{composite}}(x, \varepsilon) = e^{-1} - e^{-x} + (e^{-1} - 1)\left(e^{\frac{x-1}{\varepsilon}} - 1\right). \quad (3.23)$$

Now, the one-term SCEM approximation can be determined substituting the approximation

$$y_0^{\text{scecm}}(x, \bar{x}, \varepsilon) = y_0(x, \varepsilon) + \Psi_0(\bar{x}, \varepsilon) = 1 - e^{-x} + \Psi_0(\bar{x}, \varepsilon) \quad (3.24)$$

into the original problem (3.15):

$$\begin{aligned} -\varepsilon \frac{d^2}{dx^2} \left(1 - e^{-x} + \Psi_0(\bar{x}, \varepsilon)\right) + \frac{d}{dx} \left(1 - e^{-x} + \Psi_0(\bar{x}, \varepsilon)\right) + \left(1 - e^{-x} + \Psi_0(\bar{x}, \varepsilon)\right) &= 1 \\ -\varepsilon \left(-e^{-x} + \frac{\Psi_0''(\bar{x}, \varepsilon)}{\varepsilon^2}\right) + \left(e^{-x} + \frac{\Psi_0'(\bar{x}, \varepsilon)}{\varepsilon}\right) + \left(1 - e^{-x} + \Psi_0(\bar{x}, \varepsilon)\right) &= 1 \\ \varepsilon^2 e^{-x} - \Psi_0''(\bar{x}, \varepsilon) + \Psi_0'(\bar{x}, \varepsilon) + \varepsilon^2 \Psi_0(\bar{x}, \varepsilon) &= 0. \end{aligned} \quad (3.25)$$

In order to obtain first term of SCEM approximation, letting $\varepsilon = 0$, the first complementary SCEM approximation $\Psi_0(\bar{x}, \varepsilon)$ is obtained as the solution of the following two-point boundary value problem

$$\Psi_0''(\bar{x}, \varepsilon) - \Psi_0'(\bar{x}, \varepsilon) = 0, \quad \Psi_0\left(0, \frac{-1}{\varepsilon}, \varepsilon\right) = 0, \quad \Psi_0(1, 0, \varepsilon) = e^{-1} - 1. \quad (3.26)$$

Finally, the first SCEM and MMAE approximation are found as follows

$$y_0^{\text{scecm}}(x, \bar{x}, \varepsilon) = 1 - e^{-x} + \left(e^{\frac{x-1}{\varepsilon}} - e^{\frac{-1}{\varepsilon}}\right) \left(\frac{1 - \frac{1}{e}}{e^{\frac{-1}{\varepsilon}} - 1}\right), \quad (3.27)$$

ε	L^2 error in MMAE	L^2 error in SCEM
0.0050	0.138999385861808	0.138999385861808
0.0070	0.192846875716269	0.192846875716269
0.0100	0.271762063576098	0.271762063576098
0.0500	1.121307087312208	1.121307202107427
0.0700	1.413966007685341	1.414000101581222
0.1000	1.703813183303206	1.706213659924062
0.2500	1.129307299656383	1.882651704001525
0.5000	4.188478595061564	2.177262694969191
0.7000	7.946036759774072	2.632927334749375
0.8000	9.570235541695139	2.846650013414474
0.9000	11.023403578129694	3.041045798307327
1.0000	12.321456684111308	3.215647240288718

Table 3: L^2 -norm errors in MMAE and SCEM approximations for Example 2

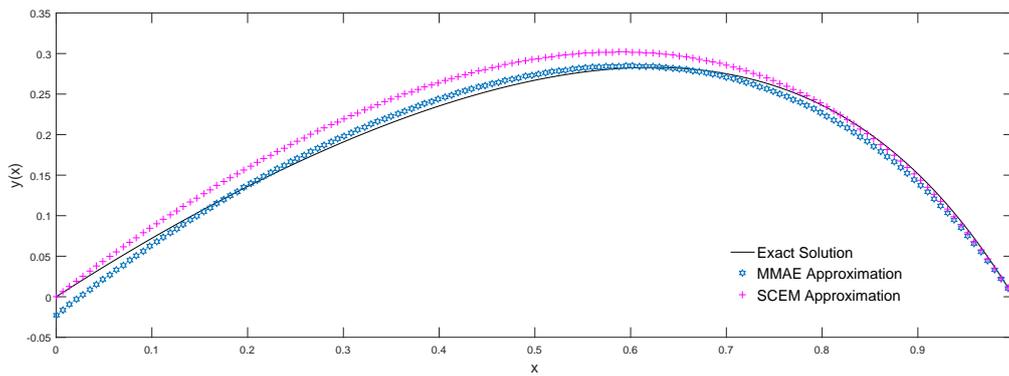


Figure 6: Comparison of results for Example 2, $\varepsilon = 0.3$

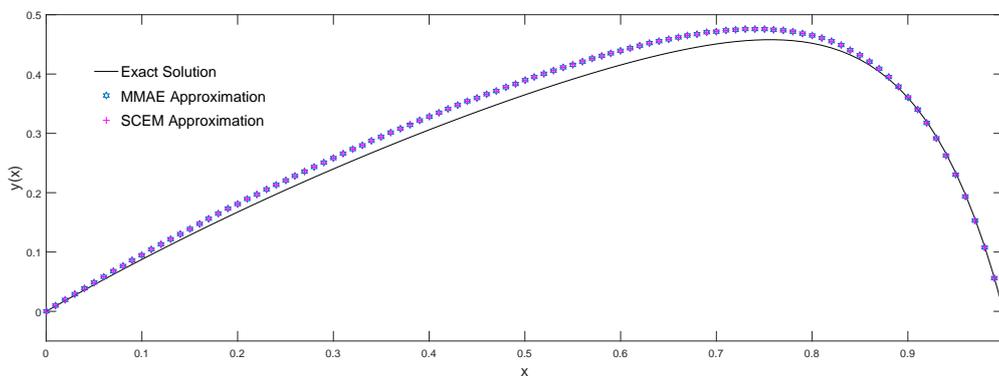
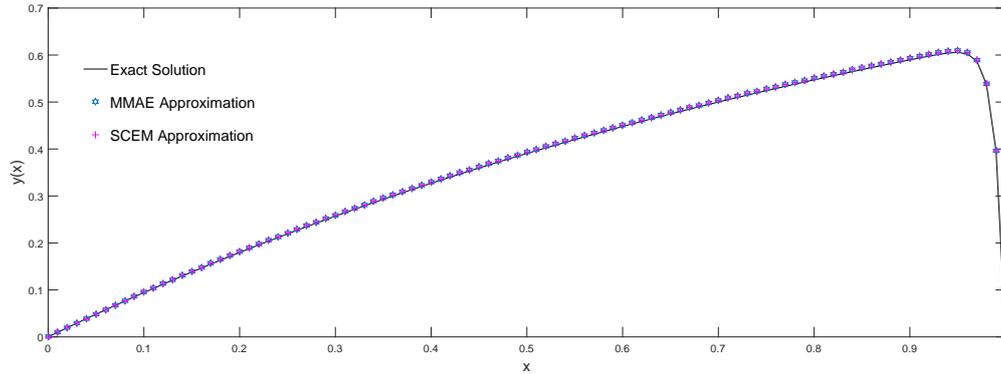


Figure 7: Comparison of results for Example 2, $\varepsilon = 0.1$

Figure 8: Comparison of results for Example 2, $\varepsilon = 0.01$

$$y_{\text{composite}}(x, \varepsilon) = e^{-1} - e^{-x} + (e^{-1} - 1) \left(e^{\frac{x-1}{\varepsilon}} - 1 \right). \quad (3.28)$$

In Fig.6 - Fig.8, comparisons of approximations that are generated by SCEM and MMAE are given for $\varepsilon = 0.3$, $\varepsilon = 0.1$ and $\varepsilon = 0.01$, respectively. It can be pointed out that for $\varepsilon = 0.3$, MMAE generates relatively more accurate approximation. As it can be observed in Table 3, for more smaller ε values, SCEM and MMAE approximations exactly overlap. In general manner, MMAE approximations are a bit more accurate than SCEM approximations, but for increasing ε values, especially for larger values than $\varepsilon = 0.5$, MMAE approximations are unacceptable.

4. Conclusion

In this paper, the well-known method MMAE and relatively new one SCEM are compared for solving singularly perturbed linear problems. An illustrative example is given to demonstrate all the steps of both methods in detail. Two non-homogeneous equations that has a left-end boundary layer and that has right-end boundary layer are provided respectively in order to analyze different cases. It is observed that SCEM gives more accurate approximations than MMAE approximations for homogeneous problems since non-homogeneous parts lead loss in captured terms during the balancing process. On the other hand, it is observed that MMAE approximations are a bit more accurate than those that are obtained by SCEM for solving non-homogeneous problems. Although only one-term approximations are proposed, SCEM and MMAE gives highly accurate approximations for moderately small ε values. It is observed that if ε is not kept sufficiently small, both SCEM and MMAE approximations differ from the exact solutions. The comparisons show that SCEM is an effective and flexible alternative method for solving singularly perturbed, especially for homogeneous problems. Furthermore, the boundary conditions are satisfied exactly and any matching process is not required contrary to MMAE.

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